

Weak Turbulence Plasma Induced by Two-Scale Homogenization

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Abstract

This paper is devoted to the homogenization of the quasilinear theory of the plasma turbulence described by the Vlasov-Poisson system. It is shown that the homogenization limit, in the sense of two-scale limit, of the distribution function satisfies the linear Vlasov-Poisson equations. Moreover, the limit distribution function can be decomposed into the mean and the fluctuation parts and the mean part (the equilibrium distribution function) is shown to be the solution of the nonlocal quasilinear velocity-space diffusion equation. We also investigate the Landau damping from the point of view of homogenization through the two-scale limit.

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1 Introduction

There are many situations in plasma, at a given instant where electrons are slightly displaced from their equilibrium positions, that bring about a consequence of the internal electric fields produced by charge separation. Electrons can bounce repeatedly back and forth as they interact with the self-consistent electric fields, and therefore these fields can be viewed as highly periodic electric fields. Furthermore, the dynamics itself can cause the so-called quasilinear diffusion motion of the electrons and the phenomenon of the interaction between particles and electric field. In order to focus on the most essential features of this theory we consider the homogenization problem of the one-dimensional, uniform, and unmagnetized, one-species plasma described by the Vlasov-Poisson system

$$\partial_t f^\epsilon(x, v, t) + \epsilon v \cdot \partial_x f^\epsilon(x, v, t) - \frac{e}{m} E^\epsilon(x, t) \partial_v f^\epsilon(x, v, t) = 0, \quad (1.1)$$

$$\epsilon \partial_x E^\epsilon(x, t) = -4\pi e \rho^\epsilon(x, t) = -4\pi e \int_{\mathbb{R}_v} f^\epsilon(x, v, t) dv, \quad (1.2)$$

$$E^\epsilon(x, t) = \epsilon \partial_x \Phi^\epsilon(x, t), \quad (1.3)$$

where $f^\epsilon(x, v, t) = f(\frac{x}{\epsilon}, v, t)$ is the velocity distribution function of electrons at location $x \in \Omega$ which is a bounded periodic domain in the spatial space \mathbb{R}_x , traveling with velocity v in velocity space \mathbb{R}_v at time t and $v^2 f^\epsilon(x, v, t) \rightarrow 0$ as $|v| \rightarrow \infty$; e is the electron charge, $E^\epsilon = E(\frac{x}{\epsilon}, t)$ is the electric field, Φ^ϵ is the electrostatic potential, and ρ^ϵ is the space density. We also point out that the small parameter ϵ is the character describing the microscopic behaviours. The ions are also assumed to be infinitely massive, that is, ion motion will be neglected and the Vlasov-Poisson system (1.1)–(1.3) describes the nonlinear plasma waves on a uniform ion background ([24]).

Quasilinear theory is a surprising complete extension to the Landau model of the plasma wave, which shows how plasma waves can alter the equilibrium velocity distribution. It is assumed that the plasma is weakly unstable, and that the instability leads to a broad spectrum of waves that modifies the background plasma in a self-consistent way via nonlinear interaction. For the related theory of plasma, we will refer to [17], [18] and [24]. To proceed, we define the spatial average of the distribution function $f(x, v, t)$ by

$$\langle f \rangle_x(v, t) \equiv \frac{1}{L} \int f(x, v, t) dx \quad (1.4)$$

where the spatial integration is carried out over the entire length L of the one dimensional plasma. We note that the derivations from the equilibrium state that cause the inhomogeneities are very small, so that they may be considered as first-order quantities. And according to the quasilinear theory of plasma

turbulence, a good approximation closed to the equilibrium, we can write the distribution function $f^\epsilon(x, v, t)$ as the sum of a mean (the slow variable) and a fluctuation part (the fast variable) as follows;

$$f^\epsilon(x, v, t) = f_0(v, t) + f_1^\epsilon(x, v, t) \quad (1.5)$$

where the ensemble averages of the above variables are

$$\langle f^\epsilon \rangle_x(v, t) = f_0(v, t), \quad \langle f_1^\epsilon \rangle_x(v, t) = 0, \quad |f_1^\epsilon| \ll f_0. \quad (1.6)$$

Unlike linear theory, in quasilinear theory, one linearizes about a spatial averaged distribution $f_0(v, t)$, that is allowed to vary slowly in time. Substituting (1.5)–(1.6) into Vlasov-Poisson equations (1.1)–(1.2), and separating the fast and slow variables, we obtain the equation for the fluctuation distribution

$$\partial_t f_1^\epsilon(x, v, t) + \epsilon v \partial_x f_1^\epsilon(x, v, t) - \frac{e}{m} E^\epsilon(x, t) \partial_v f_0^\epsilon(x, v, t) = 0, \quad (1.7)$$

and the equation for the evolution of the mean distribution function

$$\partial_t f_0(v, t) - \frac{e}{m} E^\epsilon(x, t) \partial_v f_1^\epsilon(x, v, t) = 0. \quad (1.8)$$

That the second term of the equation (1.8) depends on ϵ makes a description that the time derivative of f_0 is a very slowly changing equilibrium. Because the system is assumed to be neutral in the equilibrium and then given by a fluctuation electric field, for the Landau model, the Poisson equation (1.2) becomes

$$\epsilon \partial_x E^\epsilon(x, t) = -4\pi e \int_{\mathbb{R}_v} f_1^\epsilon(x, v, t) dv, \quad (1.9)$$

Because of the presence of an oscillating electric field $E^\epsilon(x, t)$, the homogenization will be treated by the two scale limit on spatial variable. Eqs. (1.7), (1.8) and (1.9) constitute a closed set of equations, while Eq. (1.7) is the linear part of quasilinear theory and Eq. (1.8) is the nonlinear part. The homogenization of the linear and nonlinear parts are given respectively by the following two theorems.

Theorem 1.1 *Let $f^\epsilon(x, v, 0) = f_{\text{in}}(x, v) > 0$ be the initial distribution function satisfies $f_{\text{in}}(x, v)$ and $v^2 f_{\text{in}}(x, v)$ are bounded in $L^1 \cap L^\infty(\Omega \times \mathbb{R}_v)$, $\Phi^\epsilon(x, 0) = \Phi_{\text{in}}(x) \in H^1(\Omega)$ and $v^2 f^\epsilon(x, v, t) \rightarrow 0$ as $|v| \rightarrow \infty$ and Ω be a bounded periodic domain in \mathbb{R}_x . The sequence $\{(f^\epsilon, f_1^\epsilon, E^\epsilon)\}_\epsilon$ of solutions of (1.1), (1.7) and (1.9) converges in the two-scale limit to $(\bar{f}, \bar{f}_1, \bar{E})$ solution of the system*

$$\partial_t \bar{f}(y, v, t) + v \cdot \partial_y \bar{f}(y, v, t) - \frac{e}{m} \bar{E}(y, t) \partial_v \bar{f}(y, v, t) = 0, \quad (1.10)$$

$$\partial_y \bar{E}(y, t) = -4\pi e \int_{\mathbb{R}_v} \bar{f}_1(y, v, t) dv, \quad (1.11)$$

$$\partial_t \bar{f}_1(y, v, t) + v \cdot \partial_y \bar{f}_1(y, v, t) - \frac{e}{m} \bar{E}(y, t) \partial_v f_0(v, t) = 0. \quad (1.12)$$

Next, we have the homogenization of the nonlinear part of the quasilinear theory. The time evolution of the average distribution function f_0 is the non-local quasilinear diffusion equation and the integration shows the memory (or nonlocal) effect induced by homogenization.

Theorem 1.2 *Under the same hypothesis of Theorem 1.1, there is subsequence $\{f^\epsilon(x, v, t)\}_\epsilon$, still denoted by $\{f^\epsilon(x, v, t)\}_\epsilon$, of the solutions of the Vlasov-Poisson system (1.1)–(1.3) such that $f^\epsilon(x, v, t)$ converges weakly $*$ in $L^\infty(0, T; L^2(\Omega \times \mathbb{R}_v))$ to the equilibrium distribution function $f_0(v, t)$ solution of the nonlocal quasilinear velocity-space diffusion equation*

$$\partial_t f_0(v, t) - \partial_v \left(\int_0^t D(v, \tau, t) \partial_v f_0(v, \tau) d\tau \right) = 0, \quad (1.13)$$

where

$$D(v, \tau, t) = \frac{e^2}{m^2} \int_Y \bar{E}(y, t) \bar{E}(y - v(t - \tau), \tau) dy. \quad (1.14)$$

Moreover, if $D(v, \tau, t)$ is a Dirac δ -function

$$D(v, \tau, t) = D(v, t) \delta(t - \tau), \quad (1.15)$$

then (1.13) becomes the local quasilinear velocity-space diffusion equation

$$\partial_t f_0(v, t) - \partial_v \left(D(v, t) \partial_v f_0(v, t) \right) = 0. \quad (1.16)$$

In particular, if the limit electric field $\bar{E}(y, t)$ has the same amplitude A for every Fourier mode then the diffusion coefficient D is independent of v and is given by $D(v, t) = D(t) = A^2 \frac{e^2}{m^2} e^{2\omega_I t}$, ω_I being the constant imaginary part of the frequency.

From the scientific point of view, not so much is understood about turbulence and thus modeling of turbulence flow is an important scientific and technological problem [25, 28]. Ever since the appearance of quasilinear theory in plasma turbulence, there have been recurring controversies concerning its validity, theoretically or numerically [18, 24]. Among them, the Vlasov-Poisson equations play a very important role. Indeed, the Vlasov-Poisson

equations form the simplest system of equations which describe the microscopic properties of a plasma. Therefore, this system is the starting point for any prediction about the microscopic properties of a plasma. Homogenization is a useful concept in understanding the quasilinear plasma turbulence. Thus the mathematical analysis of the Vlasov-Poisson system and the related kinetic models have been the subject of research in the last twenty years. For the existence of the solutions of the Vlasov-Poisson system we refer to [29, 30]. The homogenization of the Vlasov-Poisson system with a strong external magnetic field is studied by Frénod and Sonnendrücker in [10] and various asymptotic limits are discussed by Golse and Saint-Raymont [11]. Similar results concerning the semiconductor Boltzmann-Poisson system are studied in [5, 6, 20] and the transport equation is referred to [4, 8, 12]. For homogenization tackling memory effects the readers are referred to [1, 4, 13-16, 23, 25-28].

This paper is organized as follows: In section 2, we recall the useful properties of the two-scale convergence and prove the basic estimates. These estimates are essential to derive the homogenization limit. In section 3, we apply the uniform bounds obtained in section 2 to prove the Theorems 1.1 and 1.2. The final section is devoted to the Landau damping through the two-scale limit. In particular, the explicit formula is obtained for the normalized Maxwellian initial distribution.

2 Basic a-priori estimates

The two-scale convergence was introduced by G. Nguetseng [21] and G. Allaire [2] as an efficient tool to study the homogenization problem. It is an alternative approach to the energy method of Tartar (see [7] and references therein). In particular, in applications there are homogenization problems where the solutions do not have classical limit and the weak limit cannot be viewed as a satisfactory approximation of the solution, the asymptotic behavior of the solution can be characterized by so-called *two-scale limit* (see [3, 12, 19, 22] for detail and applications).

We denote by $C_{\#}^{\infty}(Y)$ the space of infinitely differentiable functions in \mathbb{R}_x where the subindex x indicates the x -variable dependency, that are periodic of period $Y = [0, 1)$. For $p > 1$ and an open subset $\Omega \subset \mathbb{R}$, $L^p(\Omega; C_{\#}^{\infty}(Y))$ is the space of measurable functions $v(x, y)$ on $\Omega \times Y$ such that for almost all x the function $y \rightarrow v(x, y)$ belongs to $C_{\#}^{\infty}(Y)$ with

$$\int_{\Omega} (\sup_Y |v(x, y)|)^p dx < \infty.$$

A bounded sequence $\{u^{\epsilon}\}_{\epsilon}$ of $L^p(\Omega)$ is said (weakly) two-scale converge to

$u(x, y) \in L^p(\Omega \times Y)$ if and only if

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} u^{\epsilon}(x) \psi\left(x, \frac{x}{\epsilon}\right) dx = \int_{\Omega} \int_Y u(x, y) \psi(x, y) dy dx \quad (2.1)$$

for any function $\psi(x, y) \in \mathcal{D}(\Omega; C_{\#}^{\infty}(Y))$ that is Y -periodic with respect to the second argument. This definition is justified by the following compactness theorem.

Theorem 2.1 *Let $\psi(x, x/\epsilon)$ be measurable in Ω and $\psi(x, y) \in L^p(\Omega; C_{\#}^{\infty}(Y))$, $1 < p < \infty$ then for $\epsilon > 0$ we have*

$$\left\| \psi\left(x, \frac{x}{\epsilon}\right) \right\|_{L^p(\Omega)} \leq \|\psi(x, y)\|_{L^p(\Omega; C_{\#}^{\infty}(Y))} \equiv \left[\int_{\Omega} \sup_{y \in Y} |\psi(x, y)|^p dx \right]^{\frac{1}{p}} \quad (2.2)$$

Moreover, if $\psi(x, y) \in L^p(\Omega; C_{\#}^{\infty}(Y))$ then

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \psi^p\left(x, \frac{x}{\epsilon}\right) dx = \int_{\Omega} \int_Y \psi^p(x, y) dy dx \quad (2.3)$$

and $\psi\left(x, \frac{x}{\epsilon}\right)$ two-scale converges to $\psi(x, y)$.

The proof is similar to the L^2 case as given by Allaire in [2] with modification (see also [3, 12]). Therefore, the proof is omitted. We now focus our attention to derive the priori estimates that are available for the Vlasov equation. First of all, we notice that its solution $f^{\epsilon}(x, v, t)$ satisfies the following estimate.

Lemma 2.1 *Under assumptions (1.1)–(1.4), there exists a constant C independent of ϵ such that the solution f^{ϵ} of the Vlasov-Poisson system satisfies*

$$\|f^{\epsilon}\|_{L^{\infty}(0, T; L^2(\Omega \times \mathbb{R}_v))} \leq C \quad (2.4)$$

Proof. Multiplying the Vlasov equation (1.1) by f^{ϵ} and integrating over $\Omega \times \mathbb{R}_v$ we obtain the following equality

$$\begin{aligned} & \frac{1}{2} \iint_{\Omega \times \mathbb{R}_v} \partial_t (f^{\epsilon}(x, v, t))^2 dx dv + \epsilon v \frac{1}{2} \iint_{\Omega \times \mathbb{R}_v} \partial_x (f^{\epsilon}(x, v, t))^2 dx dv \\ & - \frac{1}{2} \frac{e}{m} \iint_{\Omega \times \mathbb{R}_v} E\left(\frac{x}{\epsilon}, t\right) \partial_v (f^{\epsilon}(x, v, t))^2 dx dv = 0. \end{aligned} \quad (2.5)$$

The second and third integrals vanish after integration by part. Hence, we get

$$\frac{d}{dt} \int_{\Omega \times \mathbb{R}_v} (f^{\epsilon}(x, v, t))^2 dx dv = 0. \quad (2.6)$$

The L^2 norm of f^ϵ is conserved and (2.4) follows immediately because $f_{in} \in L^2(\Omega \times \mathbb{R}_v)$. This completes the proof. ■

We deduce from Lemma 2.1 that there exists $f \in L^\infty(0, T; L^2(\Omega \times \mathbb{R}_v))$ such that, up to subsequence,

$$f^\epsilon \rightharpoonup f \text{ in } L^\infty(0, T; L^2(\Omega \times \mathbb{R}_v)) \text{ weak} - * \quad (2.7)$$

The homogenization of the Vlasov-Poisson equation relies on the macroscopic averages such as the density and current. Integrating Eq.(1.1) over \mathbb{R}_v , we get

$$\partial_t \int_{\mathbb{R}_v} f^\epsilon(x, v, t) dv + \epsilon \partial_x \int_{\mathbb{R}_v} v f^\epsilon(x, v, t) dv - \int_{\mathbb{R}_v} \frac{e}{m} E^\epsilon(x, t) \partial_v f^\epsilon(v, t) dv = 0. \quad (2.8)$$

Then integration by parts, we derive the charge continuity equation

$$\partial_t \rho^\epsilon(x, t) + \epsilon \partial_x J^\epsilon(x, t) = 0, \quad (2.9)$$

where ρ^ϵ is the macroscopic density

$$\begin{aligned} \rho^\epsilon(x, t) &= \rho_0(t) + \rho_1^\epsilon(x, t) \\ &= \int_{\mathbb{R}_v} f_0(v, t) dv + \int_{\mathbb{R}_v} f_1^\epsilon(x, v, t) dv. \end{aligned} \quad (2.10)$$

We note that $\rho_1^\epsilon(x, t) = -\frac{\epsilon}{4\pi e} \partial_x E(x, t)$ and J^ϵ is the macroscopic current density

$$\begin{aligned} J^\epsilon(x, t) &= J_0(t) + J_1^\epsilon(x, t) \\ &= \int_{\mathbb{R}_v} v f_0(v, t) dv + \int_{\mathbb{R}_v} v f_1^\epsilon(x, v, t) dv. \end{aligned} \quad (2.11)$$

Integrating Eq. (1.8) over \mathbb{R}_v we derive ρ_0 is independent on t and eq.(2.11) means the current J_0 is independent on x . We can further rewrite charge continuity equation (2.9) as

$$\partial_t \rho_1^\epsilon(x, t) + \epsilon \partial_x J_1^\epsilon(x, t) = 0. \quad (2.12)$$

Employing (2.12) and integrating by part we obtain

$$\begin{aligned} \int_{\Omega} J_1^\epsilon(x, t) E^\epsilon(x, t) dx &= \epsilon \int_{\Omega} J_1^\epsilon(x, t) \partial_x \Phi^\epsilon(x, t) dx \\ &= -\epsilon \int_{\Omega} \partial_x J_1^\epsilon(x, t) \Phi^\epsilon(x, t) dx \\ &= \int_{\Omega} \partial_t \rho_1^\epsilon(x, t) \Phi^\epsilon(x, t) dx. \end{aligned} \quad (2.13)$$

Thus, by Poisson equation (1.9) and integrating by part we have

$$\begin{aligned} \int_{\Omega} \partial_t \rho_1^\epsilon(x, t) \Phi^\epsilon(x, t) dx &= \frac{\epsilon^2}{4\pi e} \int_{\Omega} (\partial_t \partial_x \Phi^\epsilon(x, t)) \partial_x \Phi^\epsilon(x, t) dx \\ &= \frac{\epsilon^2}{8\pi e} \partial_t \int_{\Omega} (\partial_x \Phi^\epsilon(x, t))^2 dx. \end{aligned} \quad (2.14)$$

Moreover, we multiply the Vlasov equation (1.1) by v^2 and integrate over $\mathbb{R}_v \times \Omega$ to obtain

$$\partial_t \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx - \frac{e}{m} \iint_{\mathbb{R}_v \times \Omega} E^\epsilon(x, t) v^2 \partial_v f^\epsilon(x, v, t) dv dx = 0, \quad (2.15)$$

then after integration by parts, (2.15) becomes

$$\partial_t \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx + 2 \frac{e}{m} \iint_{\mathbb{R}_v \times \Omega} E^\epsilon(x, t) v f^\epsilon(x, v, t) dv dx = 0. \quad (2.16)$$

Since $f^\epsilon = f_0 + f_1^\epsilon$, we can rewrite (2.16) as

$$\begin{aligned} \partial_t \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx &+ 2 \frac{e}{m} \iint_{\mathbb{R}_v \times \Omega} v E^\epsilon(x, t) f_0(v, t) dv dx \\ &+ 2 \frac{e}{m} \iint_{\mathbb{R}_v \times \Omega} v E^\epsilon(x, t) f_1^\epsilon(x, v, t) dv dx = 0. \end{aligned} \quad (2.17)$$

Because of the mean zero of the electric field E^ϵ , the second term of (2.17) vanishes and it becomes

$$\partial_t \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx + 2 \frac{e}{m} \int_{\Omega} E^\epsilon(x, t) J_1^\epsilon(x, t) dx = 0. \quad (2.18)$$

Using relation (2.13), Eq. (2.18) can be further rewritten as

$$\partial_t \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx + 2 \frac{e}{m} \int_{\Omega} \partial_t \rho_1^\epsilon(x, t) \Phi^\epsilon(x, t) dx = 0. \quad (2.19)$$

Also by means of Eq.(2.14), we get

$$\partial_t \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx + \frac{1}{4\pi m} \int_{\Omega} \partial_t (\epsilon \partial_x \Phi^\epsilon(x, t))^2 dx = 0, \quad (2.20)$$

or

$$\frac{d}{dt} \left(\iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, v, t) dv dx + \frac{1}{4\pi m} \int_{\Omega} (E^\epsilon(x, t))^2 dx \right) = 0. \quad (2.21)$$

Thus we have proven the following lemma.

Lemma 2.2 *Let $v^2 f_{\text{in}}$ be bounded in $L^1(\Omega \times \mathbb{R}_v)$ and $\epsilon \Phi_{\text{in}}$ bounded in $H^1(\Omega)$, i.e., E_{in}^ϵ bounded in $L^2(\Omega)$. Then there is a constant C such that*

$$\|v^2 f^\epsilon\|_{L^\infty(0, T; L^1(\Omega \times \mathbb{R}_v))} + \|E^\epsilon\|_{L^\infty(0, T; L^2(\Omega))} \leq C. \quad (2.22)$$

We also note that

$$\begin{aligned} \iint_{\mathbb{R}_v \times \Omega} v^2 f^\epsilon(x, t) dv dx &= \iint_{\mathbb{R}_v \times \Omega} v^2 (f_0(v, t) + f_1^\epsilon(x, v, t)) dv dx \\ &= \iint_{\mathbb{R}_v \times \Omega} v^2 f_0(v, t) dv dx. \end{aligned} \quad (2.23)$$

This implies that $v^2 f_0$ is bounded in $L^\infty(0, T; L^1(\Omega \times \mathbb{R}_v))$, hence $v^2 f_1^\epsilon$ is bounded in $L^\infty(0, T; L^1(\Omega \times \mathbb{R}_v))$. Employing the conservation laws of mass and energy, we deduce that the current

$$J_1^\epsilon(x, t) = \int_{\mathbb{R}_v} v f_1^\epsilon(x, v, t) dv$$

is bounded in $L^\infty(0, T; L^1(\Omega))$. Indeed, we have

$$\int_{\Omega} |J_1^\epsilon(t)| dx \leq \left(\iint_{\mathbb{R}_v \times \Omega} |v^2 f_1^\epsilon| dv dx \right)^{1/2} \left(\iint_{\mathbb{R}_v \times \Omega} |f_1^\epsilon| dv dx \right)^{1/2} \leq C.$$

Lemma 2.3 *Let $v f_1^\epsilon(x, v, t) \in L^\infty(0, T; L^1(\Omega \times \mathbb{R}_v))$. Then*

$$\epsilon \partial_x E^\epsilon(x, t) = -4\pi e \rho_1^\epsilon(x, t) = -4\pi e \int_{\mathbb{R}_v} f_1^\epsilon(x, v, t) dv.$$

is bounded in $L^\infty(0, T; L^2(\Omega))$.

Proof. Let $R > 0$ then by Cauchy-Schwarz inequality we have

$$\begin{aligned} |\rho_1^\epsilon(x, t)| &= \int_{|v| \leq R} |f_1^\epsilon(x, v, t)| dv + \int_{|v| > R} |f_1^\epsilon(x, v, t)| dv \\ &\leq 2R \|f_1^\epsilon(x, v, t)\|_{L^\infty(\Omega \times \mathbb{R}_v)} + \frac{1}{R} \int_{\mathbb{R}_v} |v f_1^\epsilon(x, v, t)| dv. \end{aligned}$$

Choosing R such that

$$2R \|f_1^\epsilon(x, v, t)\|_{L^\infty(\Omega \times \mathbb{R}_v)} = \frac{1}{R} \int_{\mathbb{R}_v} |v f_1^\epsilon(x, v, t)| dv$$

we obtain the inequality

$$|\rho_1^\epsilon(x, t)| \leq C \left(\int_{\mathbb{R}_v} |v f_1^\epsilon(x, v, t)| dv \right)^{\frac{1}{2}},$$

for some constant C depending on $\|f_1^\epsilon(x, v, t)\|_{L^\infty(\Omega \times \mathbb{R}_v)}$. Therefore, we derive the estimate

$$\int_{\Omega} |\rho_1^\epsilon(x, t)|^2 dx \leq C \left(\int_{\mathbb{R}_v \times \Omega} |v f_1^\epsilon(x, v, t)| dv dx \right).$$

This completes the proof. ■

3 Proofs of Theorem 1.1 and Theorem 1.2

This section is devoted to the proofs of Theorem 1.1 and Theorem 1.2. The basic ideas follow from the following compactness theorems of two-scale convergence. The proofs and further descriptions are referred to [2, 7, 21].

Theorem 3.1 *For each bounded sequence $\{u^\epsilon\}_\epsilon$ in $L^p(\Omega)$, $1 < p \leq \infty$, there exists a subsequence still denoted by $\{u^\epsilon\}_\epsilon$ which two-scale converges to $u(x, y) \in L^p(\Omega \times Y)$.*

Theorem 3.2 *Let u^ϵ and $\epsilon \nabla u^\epsilon$ be two bounded sequences in $L^2(\Omega)$ and $(L^2(\Omega))^3$. Then, there exists a function $u(x, y)$ in $L^2(\Omega; H^1_\#(Y))$ such that, up to a subsequence, u^ϵ and $\epsilon \nabla u^\epsilon$ two-scale converge to $u(x, y)$ and to $\nabla_y u(x, y)$, respectively.*

We remark that Theorem 3.1 shows the well definiteness for the two-scale convergence, and which further generalizes the notion of weak convergence. Theorem 3.2 gives the properties of the derivatives, which points out that the functions can be decomposed into the divergent free part and the gradient part with divergent free part zero. From Lemma 2.1 and Lemma 2.2 we have the two-scale limiting of the Vlasov equation, and combining Lemma 2.3 with theorem 3.2 we will obtain the two-scale limiting of the Poisson equation. The detail is given as follows.

Proof of Theorem 1.1. We look at the weak formulation of Vlasov-Poisson equations. Multiplying the Vlasov equation (1.1) by the admissible function $\psi^\epsilon(x, \frac{x}{\epsilon}, v, t) = \psi(x, y, v, t)$ with compact support in (x, v, t) and periodic in $y = \frac{x}{\epsilon}$ then we have

$$\begin{aligned} & \int_{\mathcal{O}} \partial_t f^\epsilon(x, v, t) \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\ & + \int_{\mathcal{O}} \epsilon v \cdot \partial_x f^\epsilon(x, v, t) \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\ & - \frac{e}{m} \int_{\mathcal{O}} E\left(\frac{x}{\epsilon}, t\right) \partial_v f^\epsilon(x, v, t) \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv = 0, \end{aligned} \quad (3.1)$$

where $\mathcal{O} = \Omega \times (0, T) \times \mathbb{R}_v$. After integrating by parts, Eq.(3.1) can be rewritten as

$$\begin{aligned} & - \int_{\mathcal{O}} f^\epsilon(x, v, t) \partial_t \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\ & - \int_{\mathcal{O}} \epsilon v \cdot f^\epsilon(x, v, t) \partial_x \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\ & - \int_{\mathcal{O}} v \cdot f^\epsilon(x, v, t) \partial_y \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\ & + \frac{e}{m} \int_{\mathcal{O}} E\left(\frac{x}{\epsilon}, t\right) f^\epsilon(x, v, t) \partial_v \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv = 0, \end{aligned} \quad (3.2)$$

or

$$\begin{aligned}
& - \int_{\mathcal{O}} f\left(\frac{x}{\epsilon}, t, v\right) \partial_t \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\
& - \int_{\mathcal{O}} \epsilon v \cdot f\left(\frac{x}{\epsilon}, t, v\right) \partial_x \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\
& - \int_{\mathcal{O}} v \cdot f\left(\frac{x}{\epsilon}, t, v\right) \partial_y \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv \\
& + \frac{e}{m} \int_{\mathcal{O}} E\left(\frac{x}{\epsilon}, t\right) f\left(\frac{x}{\epsilon}, t, v\right) \partial_v \psi\left(x, \frac{x}{\epsilon}, t, v\right) dx dt dv = 0.
\end{aligned} \tag{3.3}$$

Passing to the two-scale limit in Eq.(3.3), yields

$$\begin{aligned}
& - \int_{\mathcal{O}} \int_Y \bar{f}(y, v, t) \partial_t \psi(x, y, t, v) dy dx dt dv \\
& - \int_{\mathcal{O}} \int_Y v \cdot \bar{f}(y, v, t) \partial_y \psi(x, y, t, v) dy dx dt dv \\
& + \frac{e}{m} \int_{\mathcal{O}} \int_Y \bar{E}(y, t) \bar{f}(y, v, t) \partial_v \psi(x, y, t, v) dy dx dt dv = 0.
\end{aligned} \tag{3.4}$$

Again integrating by parts again, we can rewrite Eq.(3.4) as

$$\begin{aligned}
& \int_{\mathcal{O}} \int_Y \partial_t \bar{f}(y, v, t) \psi(x, y, t, v) dy dx dt dv \\
& + \int_{\mathcal{O}} \int_Y v \cdot \partial_y \bar{f}(y, v, t) \psi(x, y, t, v) dy dx dt dv \\
& - \frac{e}{m} \int_{\mathcal{O}} \int_Y \bar{E}(y, t) \partial_v \bar{f}(y, v, t) \psi(x, y, t, v) dy dx dt dv = 0.
\end{aligned} \tag{3.5}$$

Therefore, we have the two-scale limit equation

$$\partial_t \bar{f}(y, v, t) + v \cdot \partial_y \bar{f}(y, v, t) - \frac{e}{m} \bar{E}(y, t) \partial_v \bar{f}(y, v, t) = 0. \tag{3.6}$$

Similarly, the two-scale limiting of Eq. (1.7) takes the form

$$\partial_t \bar{f}_1(y, v, t) + v \cdot \partial_y \bar{f}_1(y, v, t) - \frac{e}{m} \bar{E}(y, t) \partial_v f_0(v, t) = 0. \tag{3.7}$$

Note that the two-scale limit of the density function f^ϵ is of the form

$$f^\epsilon(x, v, t) = f_0(v, t) + f_1^\epsilon(x, v, t) \rightharpoonup f_0(v, t) + \bar{f}_1(y, v, t), \tag{3.8}$$

and the two-scale limiting of the Poisson equation (1.9) is

$$\partial_y \bar{E}(y, t) = -4\pi e \int_{\mathbb{R}^v} \bar{f}_1(y, v, t) dv. \tag{3.9}$$

This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. The proof will be divided into two steps.

Step 1: Combining Eqs. (3.6)–(3.7) we have

$$\partial_t f_0(v, t) - \frac{e}{m} \bar{E}(y, t) \partial_v \bar{f}_1(y, v, t) = 0. \quad (3.10)$$

Note that f_0 changes because of the product of E and \bar{f}_1 which is a second order quantity. We need to eliminate the y -variable. In fact, the quasi-linear diffusion equation describing the time evolution of the average distribution, is obtained by integrating Eq. (3.10) over Y . Since $|Y| = 1$ we have

$$\partial_t f_0(v, t) - \frac{e}{m} \int_Y \bar{E}(y, t) \partial_v \bar{f}_1(y, v, t) dy = 0. \quad (3.11)$$

Note that integrating Eq. (3.7) along the characteristic, we obtain the explicit form of $\bar{f}_1(y, v, t)$:

$$\bar{f}_1(y, v, t) = \frac{e}{m} \int_0^t \bar{E}(y - v(t - \tau), \tau) \partial_v f_0(v, \tau) d\tau. \quad (3.12)$$

Using this representation of \bar{f}_1 , Eq. (3.11) can be expressed as

$$\partial_t f_0(v, t) - \int_Y \frac{e}{m} \bar{E}(y, t) \partial_v \left[\frac{e}{m} \int_0^t \bar{E}(y - v(t - \tau), \tau) \partial_v f_0(v, \tau) d\tau \right] dy = 0. \quad (3.13)$$

Let

$$D(v, \tau, t) \equiv \frac{e^2}{m^2} \int_Y \bar{E}(y, t) \bar{E}(y - v(t - \tau), \tau) dy, \quad (3.15)$$

then Eq. (3.13) is the (nonlocal) quasilinear velocity-space diffusion equation

$$\partial_t f_0(v, t) - \partial_v \left(\int_0^t D(v, \tau, t) \partial_v f_0(v, \tau) d\tau \right) = 0. \quad (3.16)$$

Step 2: Assume the limit electric field $\bar{E}(y, t)$ has the same amplitude A for every Fourier mode

$$\bar{E}(y, t) = \sum_k A e^{-2\pi i \omega(k)t + 2\pi i y k}, \quad \omega(k) = \omega_R(k) + i\omega_I(k)$$

where \bar{E} and ω satisfy the parity conditions

$$\bar{E}(k) = \bar{E}^*(-k), \quad \omega(k) = -\omega^*(-k).$$

Here $*$ denotes the complex conjugate. The real part of ω is an odd function of k and the imaginary part is an even function of k , and the following identity holds

$$t\omega(k) + \tau\omega(-k) = t\omega(k) - \tau\omega^*(k) = (t - \tau)\omega_R(k) + i(t + \tau)\omega_I(k).$$

Then invoking the representation of the Dirac δ -function

$$\int_Y e^{2\pi iy(k+\bar{k})} dy = \delta(k + \bar{k}) \quad \text{and} \quad \sum_k e^{-2\pi ik(x-y)} = \delta(x - y)$$

and after some computations we obtain

$$\begin{aligned} & \int_Y \bar{E}(y, t) \bar{E}(y - vt + v\tau, \tau) dy \\ &= A^2 \sum_k \sum_{\bar{k}} e^{-2\pi i(\omega(k)t + \omega(\bar{k})\tau)} e^{-2\pi i(t-\tau)v\bar{k}} \left(\int_Y e^{2\pi iy(k+\bar{k})} dy \right) \\ &= A^2 \sum_k \sum_{\bar{k}} e^{-2\pi i(\omega(k)t + \omega(\bar{k})\tau)} e^{-2\pi i(t-\tau)v\bar{k}} \delta(k + \bar{k}) \\ &= A^2 \sum_k e^{-2\pi i(\omega(k)t + \omega(-k)\tau)} e^{2\pi i(t-\tau)vk} \\ &= A^2 \sum_k e^{-2\pi i(t-\tau)\omega_R(k) + 2\pi(t+\tau)\omega_I(k)} e^{2\pi i(t-\tau)vk} \\ &= A^2 e^{2\pi(t+\tau)\omega_I} \sum_k e^{-2\pi i(t-\tau)\omega_R(k)} e^{2\pi i(t-\tau)vk} \\ &= A^2 e^{4\pi\omega_I t} \delta(t - \tau). \end{aligned}$$

Note that we use the assumption that the frequency of the complex part of ω is constant, that is $\omega_I(k) = \omega_I$. Thus

$$\int_Y \frac{e^2}{m^2} \bar{E}(y, t) \bar{E}(y - vt + v\tau, \tau) dy = A^2 \frac{e^2}{m^2} e^{4\pi\omega_I t} \delta(t - \tau).$$

Hence

$$D(v, \tau, t) = \frac{e^2}{m^2} \int_Y \bar{E}(y, t) \bar{E}(y - vt + v\tau, \tau) dy = D(t) \delta(t - \tau)$$

where the quasilinear diffusion coefficient is $D(t) = A^2 \frac{e^2}{m^2} e^{4\pi\omega_I t}$. Thus we have the diffusion equation

$$\partial_t f_0(v, t) - D(t) \partial_v^2 f_0(v, t) = 0.$$

This completes the proof of Theorem 1.2. \blacksquare

4 Langmuir dispersion relation and Landau damping

In this section we investigate the Landau damping from the point of view of homogenization through the two-scale limit. First we expand the fluctuations

\bar{E} and \bar{f}_1 in terms of Fourier series

$$\bar{E}(y, t) = \sum_k \hat{E}(k, t) e^{2\pi i k y}, \quad \bar{f}_1(y, v, t) = \sum_k \hat{f}_1(k, v, t) e^{2\pi i k y} \quad (4.1)$$

where $\hat{E}(k, t)$ and $\hat{f}_1(k, v, t)$ are the Fourier coefficients of $\bar{E}(y, t)$ and $\bar{f}_1(y, v, t)$ with respect to the y -variable respectively. To determine the time dependence of $\hat{E}(k, t)$ and $\hat{f}_1(k, v, t)$ we assume

$$\begin{cases} \hat{E}(k, t) &= \hat{E}(k) e^{-2\pi i W(k, t) t} \\ \hat{f}_1(k, v, t) &= \hat{f}_1(k, v) e^{-2\pi i W(k, t) t} \end{cases} \quad (4.2)$$

where $W(k, t)$ is a time-dependent complex frequency. Thus the real electric field $\bar{E}(y, t)$ and the perturbed distribution function $\bar{f}_1(y, v, t)$ can be represented respectively as

$$\bar{E}(y, t) = \sum_k \hat{E}(k) e^{-2\pi i W(k, t) t + 2\pi i k y}, \quad (4.3)$$

$$\bar{f}_1(y, v, t) = \sum_k \hat{f}_1(k, v) e^{-2\pi i W(k, t) t + 2\pi i k y}. \quad (4.4)$$

Employing the above transforms and assuming a normal mode dependence $\sim \exp(2\pi i k y - 2\pi i W(k, t) t)$, the two-scale limit equation (3.7) can be converted into

$$-2\pi i W(k, t) \hat{f}_1(k, v) + 2\pi i k v \hat{f}_1(k, v) - \frac{e}{m} \hat{E}(k) \partial_v f_0(v, t) = 0, \quad (4.5)$$

or

$$\hat{f}_1(k, v) = \frac{e}{2\pi m} \cdot \frac{\hat{E}(k) \partial_v f_0(v, t)}{-iW(k, t) + ikv}. \quad (4.6)$$

Similarly the two-scale limit Poisson equation (3.9) becomes

$$2\pi i k \hat{E}(k) = -4\pi e \int_{\mathbb{R}_v} \hat{f}_1(k, v) dv = -\frac{4\pi e^2}{2\pi m} \int_{\mathbb{R}_v} \frac{\hat{E}(k) \partial_v f_0(v, t)}{-iW(k, t) + ikv} dv, \quad (4.7)$$

and the following dispersion relation holds

$$1 = -\frac{e^2}{\pi m k} \int_{\mathbb{R}_v} \frac{\partial_v f_0(v, t)}{W(k, t) - kv} dv. \quad (4.8)$$

Notice that, for high frequency electron plasma wave, the massive ions don't have time to respond to them, so we ignore the ion contribution; that is the equilibrium function $f_0(v, t)$ can be view of the electron density, these waves are the so called Langmuir wave.

Now we return to Eq.(4.8) which has a singularity at $v = W/k$. To calculate this integral we need the Plemelj formula

$$\lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} \frac{\phi(t)}{t - t_0 - i\tau} dt = \mathcal{P} \int_{-\infty}^{\infty} \frac{\phi(t)}{t - t_0} dt + \pi i \phi(t_0), \quad (4.9)$$

where \mathcal{P} denotes the principal value, t_0 is a point on the real axis and $\phi(t)$ is a continuous function of t . Applying the Plemelj formula (4.9) to (4.8) we have the dispersion relation

$$1 = \frac{\omega_e^2}{k^2} \left(\mathcal{P} \int_{\mathbb{R}_v} \frac{\partial_v f_0(v, t)}{v - \frac{W}{k}} dv + \pi i \partial_v f_0\left(\frac{W}{k}, t\right) \right), \quad (4.10)$$

where $\omega_e^2 = \frac{e^2}{\pi m}$. For large wavelength, i.e., k is small; we expand the denominator in powers of $\frac{kv}{W}$ to obtain the approximation

$$1 = \frac{\omega_e^2}{k^2} \left[-\frac{k}{W} \int_{\mathbb{R}_v} \partial_v f_0(v, t) \left(1 + \frac{kv}{W} + \frac{k^2 v^2}{W^2} \right) dv + \pi i \partial_v f_0\left(\frac{W}{k}, t\right) \right]. \quad (4.11)$$

When the diffusion is independent of velocity v , $D = D(t)$, the equilibrium distribution function f_0 of the velocity-space equation (1.16) is given by

$$f_0(v, t) = \frac{1}{\sqrt{4\pi b(t)}} \int_{\mathbb{R}} e^{-\frac{|v-\xi|^2}{4b(t)}} f_{in}(\xi) d\xi, \quad (4.12)$$

where $b(t) = \int_0^t D(s) ds$. Moreover, we can compute f_0 explicitly for Gaussian initial data $f_{in}(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2}$, i.e., the normalized Maxwellian with zero mean velocity. Indeed, using the facts

$$\int_{\mathbb{R}_v} f_0(v, t) dv = 1, \quad \int_{\mathbb{R}_v} v f_0(v, t) dv = 0, \quad \int_{\mathbb{R}_v} v^2 f_0(v, t) dv = 2b(t) + 1, \quad (4.13)$$

Eq. (4.11) becomes, after integration by and using (4.13),

$$1 = \frac{\omega_e^2}{W^2} + \pi i \frac{\omega_e^2}{k^2} \partial_v f_0\left(\frac{W}{k}, t\right). \quad (4.14)$$

We will discuss (4.14) separately. First, if $\lim_{v \rightarrow \frac{W}{k}} \partial_v f_0(v, t) = 0$, then $W^2 = \omega_e^2$. We have the real-valued dispersion relation W and $W \sim \omega_e$, the cold plasma approximation, then integration by part of the right hand side of (4.8) yields

$$1 = \frac{e^2}{\pi m} \int_{\mathbb{R}_v} \frac{f_0(v, t)}{(W(k, t) - kv)^2} dv. \quad (4.15)$$

Expanding the denominator of the integrand up to and including second order terms in $\frac{vk}{W}$, the equation (4.15) leads to the approximation

$$1 = \frac{e^2}{\pi m W^2(k, t)} \int_{\mathbb{R}_v} f_0(v, t) \left(1 + \frac{2kv}{W(k, t)} + \frac{3k^2v^2}{W^2(k, t)} \right) dv. \quad (4.16)$$

Employing the integral relations (4.13) for Maxwellian we obtain the dispersion relation

$$W^2(k, t) - \omega_e^2 - \frac{\omega_e^2}{W^2(k, t)} 3k^2(2b(t) + 1) = 0. \quad (4.17)$$

Using the approximation $W^2 \sim \omega_e^2$, we derived the so called *Langmuir wave dispersion relation*;

$$W^2(k, t) = \omega_e^2 + 3k^2(2b(t) + 1). \quad (4.18)$$

Second, $\partial_v f_0(\frac{W}{k}, t) \neq 0$. i.e., W is complex-valued from (4.14). Using the exact solution of f_0 given by (4.12), we deduce from (4.14)

$$1 = \frac{\omega_e^2}{W^2} - i\alpha \frac{\omega_e^2 W}{k^3} \pi e^{-\frac{(\frac{W}{k})^2}{2(1+2b(t))}}, \quad \alpha = \frac{1}{\sqrt{2\pi}(1+2b(t))^{\frac{3}{2}}}. \quad (4.19)$$

For convenience, we write (4.19) as

$$0 = D_r(W) + iD_i(W) \equiv \left(1 - \frac{\omega_e^2}{W^2} \right) + i \frac{\omega_e^2 W \pi \alpha}{k^3} e^{-\frac{W^2}{2k^2(1+2b(t))}}. \quad (4.20)$$

For convenience we write $W = \omega_e - i\gamma$. Then, up to a first approximation of (4.20), equating the coefficients of the imaginary i from the both sides we obtain

$$\gamma = D_i(\omega_e) \left(\frac{dD_r(\omega_e)}{dW} \right)^{-1} = \frac{\pi \alpha \omega_e^4}{2k^3} e^{-\frac{\omega_e^2}{2k^2(1+2b(t))}}. \quad (4.21)$$

Because of $\gamma > 0$, the distribution function is $\sim \exp(iky - i\omega_e t - \gamma t)$ and is damped in time. This is the well-known *Landau damping*. The distribution function is monotonically decreasing so as to cause Landau damping of the waves.

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