Communications in Mathematical Physics manuscript No.
(will be inserted by the editor)

An asymptotic limit of a Navier-Stokes system with capillary effects

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The date of receipt and acceptance will be inserted by the editor

Abstract: A combined incompressible and vanishing capillarity limit in the barotropic compressible Navier-Stokes equations for smooth solutions is proved. The equations are considered on the two-dimensional torus with well prepared initial data. The momentum equation contains a rotational term originating from a Coriolis force, a general Korteweg-type tensor modeling capillary effects, and a density-dependent viscosity. The limiting model is the viscous quasi-geostrophic equation for the "rotated" velocity potential. The proof of the singular limit is based on the modulated energy method with a careful choice of the correction terms.

1. Introduction

The aim of this paper is to prove a combined incompressible and vanishing capillarity limit for a two-dimensional Navier-Stokes-Korteweg system, leading to the viscous quasi-geostrophic equation. We consider the (dimensionless) mass and momentum equations for the particle density $\rho(x,t)$ and the mean velocity $u(x,t) = (u_1(x,t), u_2(x,t))$ of a fluid in the two-dimensional torus $\mathbb{T}^2$:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \mathbb{T}^2,\ t > 0, \tag{1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \rho u^\perp + \nabla p(\rho) = \operatorname{div}(K + S), \tag{2}$$

with initial conditions

$$\rho(\cdot,0) = \rho^0, \quad u(\cdot,0) = u^0 \quad \text{in } \mathbb{T}^2.$$

Here, $\rho u^\perp$ describes the Coriolis force, $u^\perp = (-u_2, u_1)$, the function $p(\rho) = \rho^\gamma/\gamma$ with $\gamma > 1$ denotes the pressure of an ideal gas obeying Boyle's law, K is the Korteweg-type tension tensor and S the viscous stress tensor.

More precisely, the free surface tension tensor is given by

$$\operatorname{div} K = \kappa_0 \rho \nabla(\sigma'(\rho)\Delta\sigma(\rho)),$$

where $\kappa_0 > 0$, which can be written in conservative form as

$$\operatorname{div} K = \kappa_0 \operatorname{div}\left(\left(\Delta S(\rho) - \frac{1}{2}S''(\rho)|\nabla\rho|^2\right)\mathbb{I} - \nabla\sigma(\rho)\otimes\nabla\sigma(\rho)\right), \qquad (3)$$

where $S'(\rho) = \rho\sigma'(\rho)^2$, $\sigma(\rho)$ is a (nonlinear) function, and $\mathbb{I}$ denotes the unit matrix in $\mathbb{R}^{2\times 2}$. For a general introduction and the physical background of Navier-Stokes-Korteweg systems, we refer to [8,13,23]. In standard Korteweg models, $\kappa(\rho) = \sigma'(\rho)^2$ defines the capillarity coefficient [13, Formula (1.29)]. In the shallow-water equation, often $\sigma(\rho) = \rho$ is used such that $\operatorname{div} K = \rho\nabla\Delta\rho$ (see, e.g., [5,31]). Bresch and Desjardins [6] employed general functions $\sigma(\rho)$ and suitable viscosities allowing for additional energy estimates (also see [24]). If $\sigma(\rho) = \sqrt{\rho}$, the third-order term can be interpreted as a quantum correction, and system (1)-(2) (without the rotational term) corresponds to the so-called quantum Navier-Stokes model, derived in [9] and analyzed in [23].

The viscous stress tensor is defined by

$$\operatorname{div} S = 2\operatorname{div}(\mu(\rho)D(u)),$$

where $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ and $\mu(\rho)$ denotes the density-dependent viscosity. Often, the viscosity in the Navier-Stokes model is assumed to be constant for the mathematical analysis [15]. Density-dependent viscosities of the form $\mu(\rho) = \rho$ were chosen in [5] and were derived, in the context of the quantum Navier-Stokes model, in [9]. The choice $\mu(\rho) = \sigma(\rho)$ allows one to exploit a certain entropy structure of the system [6].

In the special case $\sigma(\rho) = \sqrt{\rho}$ and without rotational term, the existence of global (in time) weak solutions to (4)-(5) was shown in [23]. We discuss the existence of local smooth solutions in the appendix.

Without capillary effects, system (1)-(2) reduces to the viscous shallow-water or viscous Saint-Venant equations, whose inviscid version was introduced in [33]. The viscous model was formally derived from the three-dimensional Navier-Stokes equations with a free moving boundary condition [18]. This derivation was generalized later to varying river topologies [31]. The existence of global weak or strong solutions to the Korteweg-type shallow-water equations was proved in [6,8,19,20,22] under various assumptions on the nonlinear functions. In [8], the authors obtained several existence results of weak solutions under various assumptions concerning the density dependency of the coefficients. The notion of weak solution involves test functions depending on the density; this allows one to circumvent the vacuum problem. Duan et al. [12] showed the existence of local classical solutions to the shallow-water model without capillary effects. For more details and references on the shallow-water system, we refer to the review [4].

The combined incompressible and vanishing capillarity limit studied in this work is based on the scaling $t \mapsto \varepsilon t$, $u \mapsto \varepsilon u$, $\mu(\rho) \mapsto \varepsilon\mu(\rho)$ and on the choice $\kappa_0 = \varepsilon^{2\alpha}$ $(0 < \alpha, \varepsilon < 1)$, which gives

$$\partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \quad \text{in } \mathbb{T}^2,\ t > 0, \qquad (4)$$

$$\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \frac{1}{\varepsilon}\rho_\varepsilon u_\varepsilon^\perp + \frac{1}{\varepsilon^2\gamma}\nabla(\rho_\varepsilon^\gamma) - 2\varepsilon^{2(\alpha-1)}\rho_\varepsilon\nabla(\sigma'(\rho_\varepsilon)\Delta\sigma(\rho_\varepsilon))$$
$$= 2\operatorname{div}(\mu(\rho_\varepsilon)D(u_\varepsilon)), \tag{5}$$

with the initial conditions

$$\rho_\varepsilon(\cdot,0) = \rho_\varepsilon^0, \quad u_\varepsilon(\cdot,0) = u_\varepsilon^0 \quad \text{in } \mathbb{T}^2. \tag{6}$$

The condition $\alpha < 1$ is needed to control the capillary energy; see the energy identity in Lemma 1 below.

When letting $\varepsilon \to 0$, it holds $\rho_\varepsilon \to 1$ and $\rho_\varepsilon u_\varepsilon \to \nabla^\perp\phi = (-\partial\phi/\partial x_2, \partial\phi/\partial x_1)$ in appropriate function spaces, where ϕ solves the viscous quasi-geostrophic equation [32, Chapter 6] (see Section 2 for details)

$$\partial_t(\Delta\phi - \phi) + (\nabla^\perp\phi\cdot\nabla)(\Delta\phi) = \mu(1)\Delta^2\phi \quad \text{in } \mathbb{T}^2,\ t > 0, \tag{7}$$
$$\phi(\cdot,0) = \phi^0 \quad \text{in } \mathbb{T}^2. \tag{8}$$

The objective of this paper is to make this limit rigorous. Our proof requires the (local) existence of a smooth solution to (7)-(8), which is shown in the appendix. For a proof of global weak solutions in the whole space $\mathbb{R}^2$, we refer to [16, Theorem 1.1].

Several derivations of inviscid quasi-geostrophic equations have been published; see, e.g., [10, 14, 34]. The reader is also referred to the monograph [30] for a more complete discussion of this model. The viscous equation was derived rigorously for weak solutions from the shallow-water system in [5]. The proof is essentially based on the presence of the additional viscous part $\operatorname{div}(\rho\nabla u)$ and a friction term in the momentum equation. The novelty of the present paper is that these expressions are not needed and that more general expressions can be considered. In particular, we allow for viscous terms of the type $\operatorname{div}(\mu(\rho)D(u))$, and no friction is prescribed.

In the literature, singular limits in PDEs arising in fluid mechanics have been studied extensively. The first works on the incompressible limit were obtained by Klainerman and Majda [25] and Ukai [35]. The low Mach number limit of viscous compressible flows was proved by Desjardins and Grenier [11] and by Levermore et al. [26], allowing for dispersive corrections to the stress tensor (third-order terms in the velocity and temperature). Only few works are concerned with compressible rotating fluids. Bresch et al. [7] proved the combined low Mach and low Rossby limit in the compressible Navier-Stokes equations for well-prepared initial data. The same limit for ill-prepared data was shown by Feireisl et al. [16]. Finally, let us mention the work [17] in which the Mach and Rossby numbers are proportional to certain powers of a small parameter and, depending on the powers, its limit leads to the two-dimensional incompressible Navier-Stokes system or to a linear fourth-order equation for the limiting function ϕ.

In the following, we describe our main result. In order to simplify the presentation, we assume that the nonlinearities are given by power-law functions:

$$\sigma(\rho) = \rho^s, \quad \mu(\rho) = \rho^m \quad \text{for } \rho \geq 0,$$

where $s > 0$ and $m > 0$. The exponents s and m cannot be chosen freely; we need to suppose that

$$0 < s \leq 1, \quad m = s + \frac{1}{2} \leq \frac{\gamma + 1}{2}. \tag{9}$$

This assumption includes the quantum Navier-Stokes model $s = 1/2$, $m = 1$ and the shallow-water model with $s = 1$, $m = 3/2$. Furthermore, we assume that the initial data are sufficiently regular (ensuring the local-in-time existence of smooth solutions)

$$\rho_\varepsilon^0 \in H^k(\mathbb{T}^2),\ u_\varepsilon^0 \in H^{k-1}(\mathbb{T}^2),\ \phi^0 \in H^{k+1}(\mathbb{T}^2), \quad \text{where } k > 2,$$

and that they are well prepared:

$$G_\varepsilon(\phi_\varepsilon^0) \to \phi^0,\ \varepsilon^{-1}(\rho_\varepsilon^0 - 1) \to \phi^0,\ \sqrt{\rho_\varepsilon^0}u_\varepsilon^0 \to \nabla^\perp \phi^0,\ \varepsilon^{\alpha-1}\nabla\sqrt{\rho_\varepsilon^0} \to 0 \tag{10}$$

in $L^2(\mathbb{T}^2)$ as $\varepsilon \to 0$, where $\rho_\varepsilon^0 = 1 + \varepsilon\phi_\varepsilon^0$ (this defines ϕ_ε^0),

$$G_\varepsilon(\phi_\varepsilon) = \frac{\sqrt{2}}{\varepsilon}\operatorname{sign}(\phi_\varepsilon)\sqrt{h(1 + \varepsilon\phi_\varepsilon)}, \quad \rho_\varepsilon = 1 + \varepsilon\phi_\varepsilon, \tag{11}$$

and the internal energy $h(\rho)$ is defined by $h''(\rho) = p'(\rho)/\rho = \rho^{\gamma-2}$ and $h(1) = h'(1) = 0$ (see (13) for an explicit expression). Note that the convergence $\varepsilon^{-1}(\rho_\varepsilon^0 - 1) \to \phi^0$ in $L^2(\mathbb{T}^2)$ implies that $G_\varepsilon(\phi_\varepsilon^0) \to \phi^0$ in $L^1(\mathbb{T}^2)$ if ρ_ε^0 is bounded in $L^\infty(\mathbb{T}^2)$ (see (17)).

Theorem 1. *Let $0 < \alpha < 1$ and $\gamma > 1$. We suppose that (9) holds and that the initial data satisfy (10). Furthermore, let $(\rho_\varepsilon, u_\varepsilon)$ be the classical solution to (4)-(6) and let ϕ be the classical solution to (7)-(8), both on the time interval $(0, T)$. Then, as $\varepsilon \to 0$,*

$$\rho_\varepsilon \to 1 \quad \text{in } L^\infty(0, T; L^\gamma(\mathbb{T}^2)),$$
$$\rho_\varepsilon u_\varepsilon \to \nabla^\perp\phi \quad \text{in } L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\mathbb{T}^2)).$$

Furthermore, if $s < \frac{1}{2}$ and $\gamma \geq 2(1 - s)$ or if $s = 1$ and $\gamma \geq 2$,

$$\rho_\varepsilon \to 1 \quad \text{in } L^\infty(0, T; L^p(\mathbb{T}^2)),$$
$$\rho_\varepsilon u_\varepsilon \to \nabla^\perp\phi \quad \text{in } L^\infty(0, T; L^q(\mathbb{T}^2)),$$

for all $1 \leq p < \infty$ and $1 \leq q < 2$.

The proof is based on the modulated energy method, first introduced by Brenier in a kinetic context [2] and later extended to various models, e.g. [1,3, 28]. The idea of the method is to estimate, through its time derivative, a suitable modification of the energy by introducing in the energy the solution of the limit equation. We suggest the following form of the modulated energy:

$$\begin{aligned} H_\varepsilon(t) = &\int_{\mathbb{T}^2} \left(\frac{\rho_\varepsilon}{2}|u_\varepsilon - \nabla^\perp\phi|^2 + \frac{1}{2}|G_\varepsilon(\phi_\varepsilon) - \phi|^2 + 2\varepsilon^{2(\alpha-1)}|\nabla\sigma(\rho_\varepsilon)|^2 \right) dx \\ &+ 2\int_0^t \int_{\mathbb{T}^2} \mu(\rho_\varepsilon)|D(u_\varepsilon) - D(\nabla^\perp\phi)|^2 dx, \end{aligned} \tag{12}$$

These terms express the differences of the kinetic, internal, and Korteweg energies as well as the viscosity. Differentiating the modulated energy with respect to time and employing the evolution equations, elaborated computations lead to the inequality

$$H_\varepsilon(t) \le C \int_0^t H_\varepsilon(s)ds + o(1), \quad t > 0,$$

where $o(1)$ denotes terms vanishing in the limit $\varepsilon \to 0$, uniformly in time. The Gronwall lemma then implies the result.

The paper is organized as follows. In Section 2, we derive the energy identities for the shallow-water system and the quasi-geostrophic equation and give a formal derivation of the latter model from the former one. Theorem 1 is proved in Section 3. In the appendix, we discuss the existence of local smooth solutions to (4)-(5) and give an existence proof for local smooth solutions to (7)-(8).

2. Auxiliary results

In this section, we derive the energy estimates for (4)-(5) and derive formally the quasi-geostrophic equation (7). Based on the definition $h''(\rho) = p'(\rho)/\rho$, $h(1) = h'(1) = 0$, we can give an explicit formula for this function:

$$h(\rho) = \frac{1}{\gamma(\gamma-1)}\big(\rho^\gamma - 1 - \gamma(\rho-1)\big), \quad \rho \ge 0. \tag{13}$$

The energy identity for (4)-(5) is given as follows.

Lemma 1. *Let $(\rho_\varepsilon, u_\varepsilon)$ be a smooth solution to (4)-(6) on $(0,T)$. Then the energy identity*

$$\frac{dE_\varepsilon}{dt} + D_\varepsilon = 0, \quad t \in (0,T),$$

holds, where the energy E_ε and energy dissipation D_ε are defined by, respectively,

$$E_\varepsilon = \int_{\mathbb{T}^2} \left(\frac{1}{\varepsilon^2} h(\rho_\varepsilon) + \frac{1}{2}\rho_\varepsilon |u_\varepsilon|^2 + 2\varepsilon^{2(\alpha-1)} |\nabla\sigma(\rho_\varepsilon)|^2 \right) dx,$$
$$D_\varepsilon = 2\int_{\mathbb{T}^2} \mu(\rho_\varepsilon)|D(u_\varepsilon)|^2 dx.$$

Proof. Multiply (4) by $\varepsilon^{-2}h'(\rho_\varepsilon) - \frac{1}{2}|u_\varepsilon|^2 - 2\varepsilon^{2(\alpha-1)}\sigma'(\rho_\varepsilon)\Delta\sigma(\rho_\varepsilon)$, integrate over $\mathbb{T}^2$, and then integrate by parts:

$$0 = \int_{\mathbb{T}}^2 \Big(\frac{1}{\varepsilon^2}\partial_t h(\rho_\varepsilon) - \frac{1}{\varepsilon^2} h''(\rho_\varepsilon)\nabla\rho_\varepsilon \cdot (\rho_\varepsilon u_\varepsilon) - \frac{1}{2}|u_\varepsilon|^2 \partial_t \rho_\varepsilon + \rho_\varepsilon u_\varepsilon \cdot \nabla u_\varepsilon \cdot u_\varepsilon$$
$$+ 4\varepsilon^{2(\alpha-1)}\nabla\sigma(\rho_\varepsilon)\cdot\nabla\partial_t\sigma(\rho_\varepsilon) - 2\varepsilon^{2(\alpha-1)} \operatorname{div}(\rho_\varepsilon u_\varepsilon)\sigma'(\rho_\varepsilon)\Delta\sigma(\rho_\varepsilon)\Big)dx.$$

Multiplying (5) by u_ε and integrating over $\mathbb{T}^2$ gives, since $u_\varepsilon^\perp \cdot u_\varepsilon = 0$,

$$0 = \int_{\mathbb{T}^2} \Big(\partial_t(\rho u_\varepsilon)\cdot u_\varepsilon - \rho_\varepsilon (u_\varepsilon \otimes u_\varepsilon) : \nabla u_\varepsilon + \frac{1}{\varepsilon^2}\rho_\varepsilon^{\gamma-1}\nabla\rho_\varepsilon \cdot u_\varepsilon$$
$$+ 2\varepsilon^{2(\alpha-1)}\sigma'(\rho_\varepsilon)\Delta\sigma(\rho_\varepsilon)\operatorname{div}(\rho_\varepsilon u_\varepsilon) - 2\mu(\rho_\varepsilon)D(u_\varepsilon) : \nabla u_\varepsilon \Big)dx,$$

where ":" means summation over both matrix indices. Observing that h satisfies $h''(\rho_\varepsilon) = \rho_\varepsilon^{\gamma-2}$ and using the identity $D(u_\varepsilon) : \nabla u_\varepsilon = |D(u_\varepsilon)|^2$, the sum of the above two equations becomes

$$\frac{d}{dt}\int_{\mathbb{T}^2}\left(\frac{1}{\varepsilon^2}h(\rho_\varepsilon) + \frac{1}{2}\rho_\varepsilon|u_\varepsilon|^2 + 2\varepsilon^{2(\alpha-1)}|\nabla\sigma(\rho_\varepsilon)|^2\right)dx + 2\int_{\mathbb{T}^2}\mu(\rho_\varepsilon)|D(u_\varepsilon)|^2 dx = 0,$$

which proves the lemma.

A consequence of the energy identity is the following estimate.

Lemma 2. *Let $(\rho_\varepsilon, u_\varepsilon)$ be a smooth solution to* (4)-(6) *on* $(0, T)$*. Then there exists $C > 0$ such that for all $0 < \varepsilon < 1$,*

$$\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))} \le C\varepsilon^{\min\{1,2/\gamma\}} \quad \text{if } \gamma > 1, \tag{14}$$

$$\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \le C\varepsilon \quad \text{if } \gamma \ge 2. \tag{15}$$

Proof. If $\gamma = 2$, $h(\rho) = \frac{1}{2}(\rho-1)^2$, and the result follows immediately from Lemma 1. Let $\gamma > 2$. We claim that $h(\rho) \ge |\rho-1|^\gamma/(\gamma(\gamma-1))$ for $\rho \ge 0$. Then the result follows again from the energy identity. Indeed, the function $f(\rho) = \rho^\gamma - 1 - \gamma(\rho-1) - |\rho-1|^\gamma$ is convex in $(\frac{1}{2}, \infty)$ and concave in $(0, \frac{1}{2})$. Since the values $f(0) = \gamma - 2$ and $f(\frac{1}{2}) = \gamma/2 - 1$ are positive, $f \ge 0$ on $[0, \frac{1}{2}]$. Furthermore, $f(1) = f'(1) = 0$ which implies, together with the convexity, that $f \ge 0$ in $[\frac{1}{2}, \infty)$, proving the claim. Finally, let $\gamma < 2$. By [29, p. 591], $h(\rho) \ge c_R|\rho-1|^2$ for $\rho \le R$ and $h(\rho) \ge c_R|\rho-1|^\gamma$ for $\rho > R$, for some $c_R > 0$ and $R > 0$. Hence, using Hölder's inequality and $\gamma < 2$,

$$\begin{aligned}\|\rho_\varepsilon - 1\|_{L^\gamma(\mathbb{T}^2)}^\gamma &\le C\left(\int_{\{\rho_\varepsilon \le R\}}|\rho_\varepsilon - 1|^2 dx\right)^{\gamma/2} + \int_{\{\rho_\varepsilon > R\}}|\rho_\varepsilon - 1|^\gamma dx \\ &\le C\left(\int_{\{\rho_\varepsilon \le R\}} h(\rho_\varepsilon)dx\right)^{\gamma/2} + C\int_{\{\rho_\varepsilon > R\}} h(\rho_\varepsilon)dx \\ &\le C(\varepsilon^\gamma + \varepsilon^2) \le C\varepsilon^\gamma,\end{aligned}$$

where here and in the following $C > 0$ denotes a generic constant not depending on ε. Estimate (15) for $\gamma \ge 2$ follows from

$$\|\rho_\varepsilon - 1\|_{L^2(\mathbb{T}^2)}^2 = \int_{\mathbb{T}^2}(\rho_\varepsilon - 1)^2 dx \le C\int_{\mathbb{T}^2} h(\rho_\varepsilon)dx \le C\varepsilon^2,$$

which finishes the proof.

We perform the formal limit $\varepsilon \to 0$ in (4)-(5). For this, we observe that (4) can be written in terms of $\phi_\varepsilon = (\rho_\varepsilon - 1)/\varepsilon$ as follows:

$$\partial_t\phi_\varepsilon + \operatorname{div}(\phi_\varepsilon u_\varepsilon) + \frac{1}{\varepsilon}\operatorname{div} u_\varepsilon = 0.$$

We apply the operator $\operatorname{div}^\perp$ (defined by $\operatorname{div}^\perp(v_1, v_2) = -\partial v_1/\partial x_2 + \partial v_2/\partial x_1$) to (5) and observe that $\operatorname{div}^\perp(\rho_\varepsilon u_\varepsilon^\perp)/\varepsilon = \operatorname{div} u_\varepsilon/\varepsilon + \operatorname{div}(\phi_\varepsilon u_\varepsilon) = -\partial_t \phi_\varepsilon$, by the above equation. Then we find that

$$\begin{aligned}\partial_t \operatorname{div}^\perp(\rho_\varepsilon u_\varepsilon) &+ \operatorname{div}^\perp \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \partial_t \phi_\varepsilon \\ &= 2\varepsilon^{2(\alpha-1)} \operatorname{div}^\perp \big(\rho_\varepsilon \nabla(\sigma'(\rho_\varepsilon) \Delta\sigma(\rho_\varepsilon))\big) + 2 \operatorname{div}^\perp \operatorname{div}(\mu(\rho_\varepsilon) D(u_\varepsilon)). \quad (16)\end{aligned}$$

By the energy estimate, $\rho_\varepsilon \to 1$ (in $L^\infty(0, T; L^\gamma(\mathbb{T}^2))$). Assuming that $\phi_\varepsilon \to \phi$ and $u_\varepsilon \to \nabla^\perp \phi$ in suitable function spaces and employing the relations

$$\operatorname{div}^\perp \operatorname{div}(\nabla^\perp \phi \otimes \nabla^\perp \phi) = (\nabla^\perp \phi \cdot \nabla)(\Delta\phi), \quad 2 \operatorname{div}^\perp \operatorname{div}(D(\nabla^\perp \phi)) = \Delta^2 \phi,$$

the formal limit in (16) yields the limit equation (7). The initial condition reads as $\phi(\cdot, 0) = \phi^0$, where $\phi^0 = \lim_{\varepsilon\to 0} \phi_\varepsilon(\cdot, 0)$ in $\mathbb{T}^2$. The energy and the energy dissipation of (7) equal

$$E_0 = \frac{1}{2} \int_{\mathbb{T}^2} (|\nabla\phi|^2 + \phi^2) dx, \quad D_0 = 2\mu(1) \int_{\mathbb{T}^2} |D(\nabla^\perp \phi)|^2 dx.$$

Multiplying the limiting equation by ϕ and using the properties

$$\int_{\mathbb{T}^2} (\nabla^\perp \phi \cdot \nabla)(\Delta\phi)\phi dx = 0, \quad \int_{\mathbb{T}^2} (\Delta\phi)^2 dx = 2 \int_{\mathbb{T}^2} |D(\nabla^\perp \phi)|^2 dx,$$

we find the energy identity of the viscous quasi-geostrophic equation:

$$\frac{dE_0}{dt} + D_0 = 0, \quad t > 0.$$

3. Proof of Theorem 1

First, we prove the following lemma.

Lemma 3. *Let $T > 0$, $\gamma > 1$, and $0 < \alpha < 1$. Then*

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = 0 \quad \textit{uniformly in } (0, T),$$

where H_ε is defined in (12).

Proof. Using the definitions of the energy and energy dissipation as well as the relation $\frac{1}{2} G_\varepsilon(\phi_\varepsilon)^2 = \varepsilon^{-2} h(\rho_\varepsilon)$, we write

$$\begin{aligned} H_\varepsilon(t) = (E_\varepsilon + E)(t) &+ \int_0^t (D_\varepsilon + D)(s) ds + \frac{1}{2} \int_{\mathbb{T}^2} (\rho_\varepsilon - 1)|\nabla^\perp \phi|^2 dx \\ &- \int_{\mathbb{T}^2} (G_\varepsilon(\phi_\varepsilon) - \phi_\varepsilon)\phi dx - \int_{\mathbb{T}^2} \rho_\varepsilon u_\varepsilon \cdot \nabla^\perp \phi dx - \int_{\mathbb{T}^2} \phi_\varepsilon \phi dx \\ &+ 2 \int_0^t \int_{\mathbb{T}^2} (\mu(\rho_\varepsilon) - \mu(1))|D(\nabla^\perp \phi)|^2 dx ds \\ &- 4 \int_0^t \int_{\mathbb{T}^2} \mu(\rho_\varepsilon) D(u_\varepsilon) : D(\nabla^\perp \phi) dx ds \end{aligned}$$

$$= I_1 + \cdots + I_8.$$

The aim is to estimate dH_ε/dt. To this end, we treat the integrals I_j or their derivatives term by term. By the energy estimates, $\frac{d}{dt}(I_1 + I_2) = 0$. The integral I_3 cancels with a contribution originating from I_5; see below. The estimate of $I_4, \ldots, I_8$ (or their derivatives) is performed in several steps.

The key point is the estimate of the modulated potential energy I_4. We show by elementary estimations that $I_4 = o(1)$ as $\varepsilon \to 0$. The estimate of the modulated kinetic energy I_5 is new although parts of the estimates ressemble those in [28]. In the estimations of I_6, I_7, and I_8, some terms cancel with those coming from I_5. These estimates are are also new and ingenious but not difficult.

Step 1: estimate of I_4. L'Hôpital's rule shows that for $\gamma > 1$,

$$\lim_{z\to 0} \frac{h(1+z)}{z^2} = \frac{1}{2}, \quad \lim_{z\to 0} \frac{1}{z}\left(\frac{h(1+z)}{z^2} - \frac{1}{2}\right) = \frac{\gamma - 2}{6}.$$

Therefore, there exists a nonnegative function f, defined on $[0,\infty)$, such that $h(1+z) = \frac{1}{2}z^2 f(z)$ for $z \geq 0$, and a function g, defined on $[0,\infty)$, such that $f(z) - 1 = zg(z)$ for $z \geq 0$. Furthermore, the inequalities $f(z) \geq f(0) = 1$ and $|g(z)| \leq C(1 + z^{(\gamma-3)^+})$ hold, where $z^+ = \max\{0, z\}$. Finally, we claim that $f(z) = 2h(1+z)/z^2 \geq 2(1+z)^{\gamma-2}/(\gamma(\gamma-1))$ for $z \geq 0$ and $\gamma \geq 4$. Indeed, the function $w(z) = h(1+z) - z^2(1+z)^{\gamma-2}/(\gamma(\gamma-1))$ is convex in $[0,\infty)$ and $w(0) = w'(0) = 0$, which implies that $w(z) \geq 0$ in $[0,\infty)$, proving the claim. With these preparations, we can estimate the difference $G_\varepsilon(\phi_\varepsilon) - \phi_\varepsilon$ appearing in I_4:

$$\begin{aligned} |G_\varepsilon(\phi_\varepsilon) - \phi_\varepsilon| &= \left| \mathrm{sign}(\phi_\varepsilon)\left(\frac{\sqrt{2}}{\varepsilon}\sqrt{h(1+\varepsilon\phi_\varepsilon)} - |\phi_\varepsilon|\right)\right| = |\phi_\varepsilon|\left|\sqrt{f(\varepsilon\phi_\varepsilon)} - 1\right| \\ &= \frac{|\phi_\varepsilon|\,|f(\varepsilon\phi_\varepsilon) - 1|}{\sqrt{f(\varepsilon\phi_\varepsilon)} + 1} = \frac{|\phi_\varepsilon|\,|\varepsilon\phi_\varepsilon|\,|g(\varepsilon\phi_\varepsilon)|}{\sqrt{f(\varepsilon\phi_\varepsilon)} + 1}. \end{aligned}$$

In view of the bounds for f and g as well as the relation $\varepsilon\phi_\varepsilon = \rho_\varepsilon - 1$, we infer that

$$|G_\varepsilon(\phi_\varepsilon) - \phi_\varepsilon| \leq \frac{C}{\varepsilon}|\rho_\varepsilon - 1|^2 \frac{1 + \rho_\varepsilon^{(\gamma-3)^+}}{\sqrt{f(\varepsilon\phi_\varepsilon)} + 1}. \tag{17}$$

This bound allows us to estimate I_4. Indeed, if $1 < \gamma < 4$, by (14),

$$I_4(t) \leq \frac{C}{\varepsilon}\|\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}\|\rho_\varepsilon - 1\|^2_{L^\infty(0,T;L^1(\mathbb{T}^2))} \leq C\varepsilon^{2\min\{1,2/\gamma\}-1} = o(1)$$

uniformly in $(0,T)$. Here and in the following, the constant $C > 0$ depends on ϕ and its derivatives but not on ε. If $\gamma \geq 4$, we have, using the upper bound of $f(z)$ for $\gamma \geq 4$, (17), and $1 + \varepsilon\phi_\varepsilon = \rho_\varepsilon$,

$$|G_\varepsilon(\phi_\varepsilon) - \phi_\varepsilon| \leq \frac{C}{\varepsilon}|\rho_\varepsilon - 1|^2 \frac{1 + \rho_\varepsilon^{\gamma-3}}{C\rho_\varepsilon^{(\gamma-2)/2} + 1} \leq \frac{C}{\varepsilon}|\rho_\varepsilon - 1|^2\big(1 + \rho_\varepsilon^{(\gamma-3)-(\gamma-2)/2}\big).$$

We employ estimates (14)-(15) and Hölder's inequality to conclude that

$$I_4(t) \leq C\|\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}\varepsilon^{-1}\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^2(\mathbb{T}^2))}\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))}$$

$$\begin{aligned}&\times \big(1+\|\rho_\varepsilon\|^{(\gamma-4)/2}_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))}\big)\\ &\le C\varepsilon^{2/\gamma}\|\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}=o(1).\end{aligned}$$

Step 2: estimate of dI_5/dt. Inserting the momentum equation (5) and integrating by parts, it follows that

$$\begin{aligned}\frac{dI_5}{dt}&=-\int_{\mathbb{T}^2}\partial_t(\rho_\varepsilon u_\varepsilon)\cdot\nabla^\perp\phi dx-\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\partial_t\phi dx\\ &=-\int_{\mathbb{T}^2}\rho_\varepsilon(u_\varepsilon\otimes u_\varepsilon):\nabla\nabla^\perp\phi dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon^\perp\cdot\nabla^\perp\phi dx\\ &\quad+\frac{1}{\varepsilon^2\gamma}\int_{\mathbb{T}^2}\nabla\rho_\varepsilon^\gamma\cdot\nabla^\perp\phi dxd-2\varepsilon^{2(\alpha-1)}\int_{\mathbb{T}^2}\rho_\varepsilon\nabla\big(\sigma'(\rho_\varepsilon)\Delta\sigma(\rho_\varepsilon)\big)\cdot\nabla^\perp\phi dx\\ &\quad+2\int_{\mathbb{T}^2}\mu(\rho_\varepsilon)D(u_\varepsilon):\nabla\nabla^\perp\phi dx-\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\partial_t\phi dx\\ &=J_1+\cdots+J_6.\end{aligned}$$

We treat the integrals $J_1,\ldots,J_6$ term by term. The integral J_2 can be written as

$$J_2=\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla\phi dx.$$

The third integral vanishes since $\operatorname{div}\nabla^\perp=0$:

$$J_3=-\frac{1}{\varepsilon^2\gamma}\int_{\mathbb{T}^2}\rho_\varepsilon^\gamma\operatorname{div}(\nabla^\perp\phi)dx=0.$$

Using the identity (3) and $\operatorname{div}\nabla^\perp=0$, we compute

$$\begin{aligned}J_4&=\varepsilon^{2(\alpha-1)}\int_{\mathbb{T}^2}\bigg(\Big(\Delta S(\rho_\varepsilon)-\frac{1}{2}S''(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2\Big)\operatorname{div}(\nabla^\perp\phi)\\ &\qquad-(\nabla\sigma(\rho_\varepsilon)\otimes\nabla\sigma(\rho_\varepsilon)):\nabla\nabla^\perp\phi\bigg)dx\\ &\le CH_\varepsilon.\end{aligned}$$

Integration by parts and using $\operatorname{div}\nabla^\perp=0$ again yields

$$\begin{aligned}J_5&=-\int_{\mathbb{T}^2}\mu(\rho_\varepsilon)u_\varepsilon\cdot(\nabla^\perp\Delta\phi+\nabla\operatorname{div}(\nabla^\perp\phi))dx\\ &\quad-\int_{\mathbb{T}^2}\mu'(\rho_\varepsilon)(\nabla\rho_\varepsilon\otimes u_\varepsilon+u_\varepsilon\otimes\nabla\rho_\varepsilon):\nabla\nabla^\perp\phi dx\\ &=-\int_{\mathbb{T}^2}\mu(\rho_\varepsilon)u_\varepsilon\cdot\nabla^\perp\Delta\phi dx\\ &\quad-2\int_{\mathbb{T}^2}\frac{\mu'(\rho_\varepsilon)}{\sqrt{\rho_\varepsilon}\sigma'(\rho_\varepsilon)}\big(\nabla\sigma(\rho_\varepsilon)\otimes(\sqrt{\rho_\varepsilon}u_\varepsilon)+(\sqrt{\rho_\varepsilon}u_\varepsilon)\otimes\nabla\sigma(\rho_\varepsilon)\big):\nabla\nabla^\perp\phi dx.\end{aligned}$$

The assumptions on μ and σ (see (9)) yield $\mu'(\rho_\varepsilon)/(\sqrt{\rho_\varepsilon}\sigma'(\rho_\varepsilon))=\rho_\varepsilon^{m-s-1/2}$. Hence, applying the Cauchy-Schwarz inequality, the last integral is bounded from above by

$$C\|\nabla\sigma(\rho_\varepsilon)\|_{L^2(\mathbb{T}^2)}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2(\mathbb{T}^2)}\le C\varepsilon^{2(1-\alpha)}=o(1).$$

We conclude that

$$J_5 \le -\int_{\mathbb{T}^2} \mu(\rho_\varepsilon) u_\varepsilon \cdot \nabla^\perp \Delta\phi dx + o(1).$$

The integral J_6 remains unchanged. Finally, we estimate J_1. To this end, we add and substract the expression $\nabla^\perp\phi$ such that $J_1 = K_1 + \cdots + K_4$, where

$$\begin{aligned}
K_1 &= -\int_{\mathbb{T}^2} \rho_\varepsilon (u_\varepsilon - \nabla^\perp\phi) \otimes (u_\varepsilon - \nabla^\perp\phi) : \nabla\nabla^\perp\phi dx, \\
K_2 &= -\int_{\mathbb{T}^2} \rho_\varepsilon \nabla^\perp\phi \otimes u_\varepsilon : \nabla\nabla^\perp\phi dx, \\
K_3 &= -\int_{\mathbb{T}^2} \rho_\varepsilon u_\varepsilon \otimes \nabla^\perp\phi : \nabla\nabla^\perp\phi dx, \\
K_4 &= \int_{\mathbb{T}^2} \rho_\varepsilon \nabla^\perp\phi \otimes \nabla^\perp\phi : \nabla\nabla^\perp\phi dx.
\end{aligned}$$

The first integral can be bounded by the modulated energy:

$$K_1 \le C\int_{\mathbb{T}^2} \rho_\varepsilon |u_\varepsilon - \nabla^\perp\phi|^2 dx \le CH_\varepsilon.$$

A reformulation yields

$$K_2 = -\int_{\mathbb{T}^2} \rho_\varepsilon u_\varepsilon \cdot \big((\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\big) dx.$$

We employ the continuity equation (4) to find

$$\begin{aligned}
K_3 &= -\frac{1}{2}\int_{\mathbb{T}^2} \rho_\varepsilon u_\varepsilon \cdot \nabla|\nabla^\perp\phi|^2 dxd = \frac{1}{2}\int_{\mathbb{T}^2} \operatorname{div}(\rho_\varepsilon u_\varepsilon)|\nabla^\perp\phi|^2 dx \\
&= -\frac{1}{2}\int_{\mathbb{T}^2} \partial_t(\rho_\varepsilon - 1)|\nabla^\perp\phi|^2 dx \\
&= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} (\rho_\varepsilon - 1)|\nabla^\perp\phi|^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} (\rho_\varepsilon - 1)\partial_t|\nabla^\perp\phi|^2 dx \\
&= -\frac{dI_3}{dt} + o(1).
\end{aligned}$$

Finally, using again $\operatorname{div}\nabla^\perp = 0$,

$$\begin{aligned}
K_4 &= -\int_{\mathbb{T}^2} \rho_\varepsilon \big((\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\big) \cdot \nabla^\perp\phi dx \\
&= -\int_{\mathbb{T}^2} (\rho_\varepsilon - 1)\big((\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\big) \cdot \nabla^\perp\phi dx - \int_{\mathbb{T}^2} \big((\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\big) \cdot \nabla^\perp\phi dx \\
&= -\int_{\mathbb{T}^2} (\rho_\varepsilon - 1)\big((\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\big) \cdot \nabla^\perp\phi dx - \frac{1}{2}\int_{\mathbb{T}^2} \nabla^\perp\phi \cdot \nabla(|\nabla^\perp\phi|^2) dx \\
&= -\int_{\mathbb{T}^2} (\rho_\varepsilon - 1)\big((\nabla^\perp\phi \cdot \nabla)\nabla^\perp\phi\big) \cdot \nabla^\perp\phi dx + \frac{1}{2}\int_{\mathbb{T}^2} \operatorname{div}(\nabla^\perp\phi)|\nabla^\perp\phi|^2 dx \\
&= o(1).
\end{aligned}$$

In the last step, we have employed estimate (14) for $\rho_\varepsilon - 1$. Summarizing the estimates for $K_1, \ldots, K_4$, we have shown that

$$J_1 \le CH_\varepsilon - \frac{dI_3}{dt} - \int_{\mathbb{T}^2} \big((\nabla^\perp\phi\cdot\nabla)\nabla^\perp\phi\big)\cdot(\rho_\varepsilon u_\varepsilon)dx + o(1).$$

Then, summarizing the estimates for $J_1, \ldots, J_6$, we obtain

$$\begin{aligned}\frac{dI_5}{dt} &\le CH_\varepsilon - \frac{dI_3}{dt} + \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla\phi dx \\ &\quad - \int_{\mathbb{T}^2}\big((\partial_t + \nabla^\perp\phi\cdot\nabla)\nabla^\perp\phi + \mu(1)\nabla^\perp\Delta\phi\big)\cdot(\rho_\varepsilon u_\varepsilon)dx \\ &\quad - \int_{\mathbb{T}^2}\big(\mu(\rho_\varepsilon) - \mu(1)\rho_\varepsilon\big)u_\varepsilon\cdot\nabla^\perp\Delta\phi dx + o(1).\end{aligned}$$

The last integral can be estimated by employing the assumptions on μ and Hölder's inequality:

$$\int_{\mathbb{T}^2}\frac{\mu(\rho_\varepsilon)-\mu(1)\rho_\varepsilon}{\sqrt{\rho_\varepsilon}}\sqrt{\rho_\varepsilon}u_\varepsilon\cdot\nabla^\perp\Delta\phi dx \le C\|\rho_\varepsilon^{m-1/2} - \rho_\varepsilon^{1/2}\|_{L^2(\mathbb{T}^2)}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2(\mathbb{T}^2)}.$$

We claim that the first factor on the right-hand side is of order $o(1)$. To prove this statement, we consider first $\frac{1}{2} < m < 1$:

$$\begin{aligned}\|\rho_\varepsilon^{m-1/2} - \rho_\varepsilon^{1/2}\|^2_{L^2(\mathbb{T}^2)} &\le \int_{\mathbb{T}^2}\rho_\varepsilon^{2m-1}|\rho_\varepsilon - 1|^{2(1-m)}dx \\ &\le \|\rho_\varepsilon\|^{2m-1}_{L^\gamma(\mathbb{T}^2)}\|\rho_\varepsilon - 1\|^{2(1-m)}_{L^p(\mathbb{T}^2)},\end{aligned}$$

where $p = 2\gamma(1-m)/(\gamma - 2m + 1)$. The inequality $p \le \gamma$ is equivalent to $\gamma \ge 1$. Note that the Hölder inequality can be applied since we supposed that $2m - 1 \le \gamma$; see (9). Second, let $1 < m \le 2$ (the case $m = 1$ being trivial). We compute

$$\|\rho_\varepsilon^{m-1/2} - \rho_\varepsilon^{1/2}\|^2_{L^2(\mathbb{T}^2)} \le \int_{\mathbb{T}^2}\rho_\varepsilon|\rho_\varepsilon - 1|^{2(m-1)}dx \le \|\rho_\varepsilon\|_{L^\gamma(\mathbb{T}^2)}\|\rho_\varepsilon - 1\|^{2(m-1)}_{L^q(\mathbb{T}^2)},$$

where $q = 2\gamma(m-1)/(\gamma-1)$, and $q \le \gamma$ if and only if $m \le (\gamma+1)/2$. Finally, if $2 \le m \le (\gamma+1)/2$, we find that

$$\begin{aligned}\|\rho_\varepsilon^{m-1/2} - \rho_\varepsilon^{1/2}\|^2_{L^2(\mathbb{T}^2)} &\le C\int_{\mathbb{T}^2}\rho_\varepsilon(1+\rho_\varepsilon^{m-2})^2|\rho_\varepsilon - 1|^2 dx \\ &\le C(1 + \|\rho_\varepsilon\|^{2m-3}_{L^\gamma(\mathbb{T}^2)})\|\rho_\varepsilon - 1\|^2_{L^r(\mathbb{T}^2)},\end{aligned}$$

with $r = 2\gamma/(\gamma - 2m + 3)$ satisfying $r \le \gamma$ if and only if $m \le (\gamma+1)/2$. We conclude that

$$\int_{\mathbb{T}^2}\frac{\mu(\rho_\varepsilon)-\mu(1)\rho_\varepsilon}{\sqrt{\rho_\varepsilon}}\sqrt{\rho_\varepsilon}u_\varepsilon\cdot\nabla^\perp\Delta\phi dx \le C\|\rho_\varepsilon - 1\|^\beta_{L^\gamma(\mathbb{T}^2)}$$

for some $\beta > 0$, and together with (14), this shows that the integral is of order $o(1)$. Therefore,

$$\frac{dI_5}{dt} \le CH_\varepsilon - \frac{dI_3}{dt} + \frac{1}{\varepsilon}\int_{\mathbb{T}^2} \rho_\varepsilon u_\varepsilon \cdot \nabla\phi dx - \int_{\mathbb{T}^2} \big((\partial_t + \nabla^\perp\phi\cdot\nabla)\nabla^\perp\phi + \mu(1)\nabla^\perp\Delta\phi\big)\cdot(\rho_\varepsilon u_\varepsilon)dx + o(1). \tag{18}$$

Step 3: estimate of dI_6/dt. Employing (4) and (7), we can write

$$\begin{aligned}\frac{dI_6}{dt} &= -\int_{\mathbb{T}^2}\partial_t\phi_\varepsilon\phi dx - \int_{\mathbb{T}^2}\phi_\varepsilon\partial_t\phi dx \\ &= \frac{1}{\varepsilon}\int_{\mathbb{T}^2}\operatorname{div}(\rho_\varepsilon u_\varepsilon)\phi dx - \int_{\mathbb{T}^2}\big((\partial_t + \nabla^\perp\phi\cdot\nabla)(\Delta\phi) - \mu(1)\Delta^2\phi\big)\phi_\varepsilon dx \\ &= -\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla\phi dx \\ &\quad + \int_{\mathbb{T}^2}\big((\partial_t + \nabla^\perp\phi\cdot\nabla)(\nabla^\perp\phi) - \mu(1)\nabla^\perp\Delta\phi\big)\cdot\nabla^\perp\phi_\varepsilon dx. \end{aligned}\tag{19}$$

We observe that the first integral on the right-hand side cancels with the corresponding integral in (18). To deal with the second integral, we employ again the momentum equation (5). We write

$$\frac{1}{\gamma}\nabla\rho_\varepsilon^\gamma = (\gamma-1)\nabla h(\rho_\varepsilon) + \nabla(\rho_\varepsilon - 1) = (\gamma-1)\nabla h(\rho_\varepsilon) + \varepsilon\nabla\phi_\varepsilon.$$

Then, because of $(u_\varepsilon^\perp)^\perp = -u_\varepsilon$, (5) is equivalent to

$$\nabla^\perp\phi_\varepsilon = \rho_\varepsilon u_\varepsilon - \varepsilon F_\varepsilon^\perp,$$

where

$$\begin{aligned}F_\varepsilon &= \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon) + \frac{\gamma-1}{\varepsilon^2}\nabla h(\rho_\varepsilon) - 2\operatorname{div}(\mu(\rho_\varepsilon)D(u_\varepsilon)) \\ &\quad - \varepsilon^{2(\alpha-1)}\Big(\nabla\Delta S(\rho_\varepsilon) - \frac{1}{2}\nabla(S''(\rho_\varepsilon)|\rho_\varepsilon|^2) - \operatorname{div}\big(\nabla\sigma(\rho_\varepsilon)\otimes\nabla\sigma(\rho_\varepsilon)\big)\Big).\end{aligned}$$

Replacing $\nabla^\perp\phi_\varepsilon$ in the second integral in (19) by the above expression gives

$$\begin{aligned}&\int_{\mathbb{T}^2}\big((\partial_t + \nabla^\perp\phi\cdot\nabla)(\nabla^\perp\phi) - \mu(1)\nabla^\perp\Delta\phi\big)\cdot\nabla^\perp\phi_\varepsilon dx \\ &\quad = \int_{\mathbb{T}^2}\big((\partial_t + \nabla^\perp\phi\cdot\nabla)(\nabla^\perp\phi) - \mu(1)\nabla^\perp\Delta\phi\big)\cdot(\rho_\varepsilon u_\varepsilon - \varepsilon F_\varepsilon^\perp)dx.\end{aligned}$$

We claim that the integral containing $F_\varepsilon^\perp$ is bounded in an appropriate space. Indeed, let ψ be a smooth (vector-valued) test function. The first term of F_ε is written in weak form as follows:

$$\int_0^T\int_{\mathbb{T}^2}\partial_t(\rho_\varepsilon u_\varepsilon)\cdot\psi dxds = -\int_0^T\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\partial_t\psi dxds + \int_{\mathbb{T}^2}(\rho_\varepsilon u_\varepsilon)(t)\cdot\psi(t)dx$$

$$- \int_{\mathbb{T}^2} \rho_\varepsilon^0 u_\varepsilon^0 \cdot \psi(0) dx.$$

These integrals are bounded if $\rho_\varepsilon u_\varepsilon$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}^2))$. This is the case, since mass conservation and the energy estimate show that

$$\int_{\mathbb{T}^2} |\rho_\varepsilon u_\varepsilon| dx \le \frac{1}{2} \int_{\mathbb{T}^2} \rho_\varepsilon dx + \frac{1}{2} \int_{\mathbb{T}^2} \rho_\varepsilon |u_\varepsilon|^2 dx$$

is uniformly bounded in $(0,T)$. An integration by parts gives

$$\int_0^T \int_{\mathbb{T}^2} \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) \cdot \psi dx ds = - \int_0^T \int_{\mathbb{T}^2} \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \psi dx ds,$$

and this integral is uniformly bounded, by the energy estimate. Furthermore, again integrating by parts,

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^2} \left(\frac{\gamma - 1}{\varepsilon^2} \nabla h(\rho_\varepsilon) - 2 \operatorname{div}(\mu(\rho_\varepsilon) D(u_\varepsilon)) \right) \cdot \psi dx ds \\ &\qquad = - \int_0^T \int_{\mathbb{T}^2} \left(\frac{\gamma - 1}{\varepsilon^2} h(\rho_\varepsilon) \mathbb{I} - 2\mu(\rho_\varepsilon) D(u_\varepsilon) \right) : \nabla \psi dx ds, \end{aligned}$$

which is uniformly bounded since we can estimate

$$\int_0^T \int_{\mathbb{T}^2} |\mu(\rho_\varepsilon) D(u_\varepsilon)| dx ds \le \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} \mu(\rho_\varepsilon) dx ds + \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} \mu(\rho_\varepsilon) |D(u_\varepsilon)|^2 dx ds$$

and $\mu(\rho_\varepsilon) \le C(1 + \rho_\varepsilon^\gamma)$. Also the remaining terms are bounded since

$$\begin{aligned} &\varepsilon^{2(\alpha-1)} \int_0^T \int_{\mathbb{T}^2} \Big(\nabla \Delta (S(\rho_\varepsilon) - S(1)) - \frac{1}{2} \nabla (S''(\rho_\varepsilon) |\nabla \rho_\varepsilon|^2) \\ &\qquad - \operatorname{div}(\nabla \sigma(\rho_\varepsilon) \otimes \nabla \sigma(\rho_\varepsilon)) \Big) \cdot \psi dx ds \\ &\quad = - \varepsilon^{2(\alpha-1)} \int_0^T \int_{\mathbb{T}^2} \Big((S(\rho_\varepsilon) - S(1)) \Delta \operatorname{div} \psi + \frac{1}{2} S''(\rho_\varepsilon) |\nabla \rho_\varepsilon|^2 \operatorname{div} \psi \\ &\qquad - (\nabla \sigma(\rho_\varepsilon) \otimes \nabla \sigma(\rho_\varepsilon)) : \nabla \psi \Big) dx ds. \end{aligned}$$

Using the Hölder continuity of $S(z) = (s/2) z^{2s}$, $z \ge 0$, the first summand can be estimated by $C|\rho_\varepsilon - 1|^{\min\{1,2s\}}$. We infer that the corresponding integral is of order $o(1)$. We formulate the second summand as

$$\frac{1}{2} \varepsilon^{2(\alpha-1)} (2s - 1) \int_0^t \int_{\mathbb{T}^2} |\nabla \sigma(\rho_\varepsilon)|^2 \operatorname{div} \psi dx ds.$$

In view of the energy estimate, this integral as well as the third summand are uniformly bounded. This shows that

$$\begin{aligned} &\int_{\mathbb{T}^2} \big((\partial_t + \nabla^\perp \phi \cdot \nabla)(\nabla^\perp \phi) - \mu(1) \nabla^\perp \Delta \phi \big) \cdot \nabla^\perp \phi_\varepsilon dx \\ &\quad = \int_{\mathbb{T}^2} \big((\partial_t + \nabla^\perp \phi \cdot \nabla)(\nabla^\perp \phi) - \mu(1) \nabla^\perp \Delta \phi \big) \cdot (\rho_\varepsilon u_\varepsilon) dx + o(1), \end{aligned}$$

and consequently, (19) becomes

$$\frac{dI_6}{dt} = -\frac{1}{\varepsilon}\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla\phi dx + \int_{\mathbb{T}^2}\big((\partial_t + \nabla^\perp\phi\cdot\nabla)(\nabla^\perp\phi) - \mu(1)\nabla^\perp\Delta\phi\big)\cdot(\rho_\varepsilon u_\varepsilon)dx + o(1).$$

Step 4: estimate of dI_7/dt. The function μ satisfies $|\mu(z)-\mu(1)| = |z^m - 1| \le |z-1|^m$ if $m \le 1$ and $|\mu(z)-\mu(1)| \le C(1+z^{m-1})|z-1|$ if $m > 1$, for $z \ge 0$. Therefore, if $m \le 1$, taking into account (14),

$$\frac{dI_7}{dt} \le 2\|\rho_\varepsilon - 1\|^m_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))}\|D(\nabla^\perp\phi)\|^2_{L^\infty(0,T;L^{2\gamma/(\gamma-m)}(\mathbb{T}^2))} \le C\varepsilon^{m\min\{1,2/\gamma\}}.$$

Moreover, if $1 < m \le (\gamma+1)/2$, using Hölder's inequality,

$$\frac{dI_7}{dt} \le C\big(1 + \|\rho_\varepsilon\|_{L^\infty(0,T;L^{(m-1)\gamma/(\gamma-1)}(\mathbb{T}^2))}\big)\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))} \le C\varepsilon^{\min\{1,2/\gamma\}}.$$

The norm of ρ_ε is uniformly bounded since $(m-1)\gamma/(\gamma-1) \le \gamma$ is equivalent to $m \le \gamma$.

Step 5: estimate of dI_8/dt. Integration by parts yields

$$\begin{aligned}\frac{dI_8}{dt} &= \int_{\mathbb{T}^2}\mu'(\rho_\varepsilon)\nabla\rho_\varepsilon\otimes u_\varepsilon : \nabla\nabla^\perp\phi dx + 2\int_{\mathbb{T}^2}\mu(\rho_\varepsilon)u_\varepsilon\cdot\nabla^\perp\Delta\phi dx \\ &= \int_{\mathbb{T}^2}\frac{\mu'(\rho_\varepsilon)}{\sqrt{\rho_\varepsilon}\sigma'(\rho_\varepsilon)}\nabla\sigma(\rho_\varepsilon)\otimes(\sqrt{\rho_\varepsilon}u_\varepsilon) : \nabla\nabla^\perp\phi dx \\ &\quad + 2\int_{\mathbb{T}^2}(\mu(\rho_\varepsilon) - \mu(1)\rho_\varepsilon)u_\varepsilon\cdot\nabla^\perp\Delta\phi dx + 2\mu(1)\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\Delta\phi dx.\end{aligned}$$

By definition of μ and σ (see (9)), it follows that

$$\frac{dI_8}{dt} \le C\|\nabla\sigma(\rho_\varepsilon)\|_{L^2(\mathbb{T}^2)}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2(\mathbb{T}^2)} + C\|\rho_\varepsilon^{m-1/2} - \rho_\varepsilon^{1/2}\|_{L^2(\mathbb{T}^2)}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2(\mathbb{T}^2)} + 2\mu(1)\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\Delta\phi dx.$$

Because of the energy estimate, the first summand is of order $o(1)$. The second summand has been estimated in Step 2, and it has been found that it is also of order $o(1)$. This shows that

$$\frac{dI_8}{dt} \le 2\mu(1)\int_{\mathbb{T}^2}\rho_\varepsilon u_\varepsilon\cdot\nabla^\perp\Delta\psi dx + o(1).$$

Step 6: conclusion. Adding the estimates for $dI_4/dt,\ldots,dI_8/dt$, most of the integrals cancel, and we end up with

$$\frac{dH_\varepsilon}{dt} \le CH_\varepsilon + \frac{dI_4}{dt} + o(1).$$

Integrating over $(0,t)$ gives

$$H_\varepsilon(t) \leq H_\varepsilon(0) + C\int_0^t H_\varepsilon(s)ds + I_4(t) - I_4(0) + o(1).$$

By Step 1, $I_4(t) = o(1)$. Furthermore, $I_4(0) = o(1)$ by assumption. It holds that $H_\varepsilon(0) = o(1)$ since

$$\begin{aligned}\|\sqrt{\rho_\varepsilon^0}(u_\varepsilon^0 - \nabla^\perp\phi^0)\|_{L^2(\mathbb{T}^2)} &\leq \|\sqrt{\rho_\varepsilon^0}u_\varepsilon^0 - \nabla^\perp\phi^0\|_{L^2(\mathbb{T}^2)} + \|(1-\sqrt{\rho_\varepsilon^0})\nabla^\perp\phi^0\|_{L^2(\mathbb{T}^2)} \\ &\leq \|\sqrt{\rho_\varepsilon^0}u_\varepsilon^0 - \nabla^\perp\phi^0\|_{L^2(\mathbb{T}^2)} \\ &\quad + \|1-\rho_\varepsilon^0\|_{L^2(\mathbb{T}^2)}\|\nabla^\perp\phi^0\|_{L^\infty(\mathbb{T}^2)} \\ &= o(1)\end{aligned}$$

and since the initial data are well prepared. Then the Gronwall lemma implies that $H_\varepsilon(t) = o(1)$ finishing the proof.

We are now in the position to prove Theorem 1 which is a consequence of Lemma 3. We observe that by (14), $\rho_\varepsilon \to 1$ in $L^\infty(0,T;L^\gamma(\mathbb{T}^2))$ and, using the Hölder inequality and $2\gamma/(\gamma+1) < \gamma$,

$$\begin{aligned}&\|\rho_\varepsilon u_\varepsilon - \nabla^\perp\phi\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\mathbb{T}^2))} \\ &\quad\leq \|\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^{2\gamma}(\mathbb{T}^2))}\|\sqrt{\rho_\varepsilon}(u_\varepsilon - \nabla^\perp\phi)\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \\ &\quad\quad + \|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\mathbb{T}^2))}\|\nabla^\perp\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))} \\ &\quad\leq C\|\sqrt{\rho_\varepsilon}(u_\varepsilon - \nabla^\perp\phi)\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \\ &\quad\quad + C\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))}.\end{aligned} \tag{20}$$

We conclude that $\rho_\varepsilon u_\varepsilon \to \nabla^\perp\phi$ in $L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\mathbb{T}^2))$.

Next, let $\gamma \geq 2(1-s)$ and $0 < s < 1/2$. Because of assumption (9), i.e. $\gamma \geq 2s$, we have $2\gamma/(\gamma+2(1-s)) \leq \gamma$, and hence,

$$\rho_\varepsilon \to 1 \quad \text{in } L^\infty(0,T;L^{2\gamma/(\gamma+2(1-s))}(\mathbb{T}^2))$$

as $\varepsilon \to 0$. Furthermore, since $\alpha < 1$, $\nabla\sigma(\rho_\varepsilon) \to 0$ in $L^\infty(0,T;L^2(\mathbb{T}^2))$ as $\varepsilon \to 0$ and thus, by Hölder's inequality,

$$\begin{aligned}&\|\nabla(\rho_\varepsilon - 1)\|_{L^\infty(0,T;L^{2\gamma/(\gamma+2(1-s))}(\mathbb{T}^2))} \\ &\quad= \|\sigma'(\rho_\varepsilon)^{-1}\nabla\sigma(\rho_\varepsilon)\|_{L^\infty(0,T;L^{2\gamma/(\gamma+2(1-s))}(\mathbb{T}^2))} \\ &\quad\leq \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^2))}^{1-s}\|\nabla\sigma(\rho_\varepsilon)\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \to 0.\end{aligned} \tag{21}$$

We infer that $\rho_\varepsilon \to 1$ in $L^\infty(0,T;W^{1,2\gamma/(\gamma+2(1-s))}(\mathbb{T}^2))$. Because of the continuous embedding $W^{1,2\gamma/(\gamma+2(1-s))}(\mathbb{T}^2) \hookrightarrow L^{\gamma/(1-s)}(\mathbb{T}^2)$, this implies that $\rho_\varepsilon \to 1$ in $L^\infty(0,T;L^{\gamma/(1-s)}(\mathbb{T}^2))$. Since $2\gamma/(\gamma+2(1-s)^2) \leq \gamma/(1-s)$, this gives $\rho_\varepsilon \to 1$ in $L^\infty(0,T;L^{2\gamma/(\gamma+2(1-s)^2)}(\mathbb{T}^2))$. Applying the same procedure as in (21) again, we obtain

$$\begin{aligned}&\|\nabla(\rho_\varepsilon - 1)\|_{L^\infty(0,T;L^{2\gamma/(\gamma+2(1-s)^2)}(\mathbb{T}^2))} \\ &\quad\leq \|\rho_\varepsilon\|_{L^\infty(0,T;L^{\gamma/(1-s)}(\mathbb{T}^2))}^{1-s}\|\nabla\sigma(\rho_\varepsilon)\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \to 0.\end{aligned}$$

Hence, $\rho_\varepsilon \to 1$ strongly in $L^\infty(0,T;W^{1,2\gamma/(\gamma+2(1-s)^2)}(\mathbb{T}^2))$ and in $L^\infty(0,T;L^{\gamma/(1-s)^2}(\mathbb{T}^2))$. Repeating this argument, we conclude that $\rho^\varepsilon \to 1$ in $L^\infty(0,T;L^p(\mathbb{T}^2))$ for all $p<\infty$.

For the momentum, we obtain for $p \ge 1$

$$\begin{aligned}
&\|\rho_\varepsilon u_\varepsilon - \nabla^\perp\phi\|_{L^\infty(0,T;L^{2p/(p+1)}(\mathbb{T}^2))} \\
&\quad\le \|\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^{2p}(\mathbb{T}^2))}\|\sqrt{\rho_\varepsilon}(u_\varepsilon-\nabla^\perp\phi)\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \\
&\qquad + \|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^{2p/(p+1)}(\mathbb{T}^2))}\|\nabla^\perp\phi\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))} \\
&\quad\le C\|\sqrt{\rho_\varepsilon}(u_\varepsilon-\nabla^\perp\phi)\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} \\
&\qquad + C\|\rho_\varepsilon - 1\|_{L^\infty(0,T;L^p(\mathbb{T}^2))}.
\end{aligned}$$

This shows that $\rho_\varepsilon u_\varepsilon \to \nabla^\perp\phi$ in $L^\infty(0,T;L^q(\mathbb{T}^2))$ for all $q<2$.

Finally, let $\gamma \ge 2$ and $s=1$. Then $\rho_\varepsilon \to 1$ in $L^\infty(0,T;H^1(\mathbb{T}^2))$ and, by the continuous embedding $H^1(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2)$ for all $p<\infty$, also $\rho_\varepsilon \to 1$ in $L^\infty(0,T;L^p(\mathbb{T}^2))$ for all $p<\infty$. The theorem is proved.

A. Local existence of smooth solutions

The local existence of smooth solutions to the Navier-Stokes-Korteweg system (4)-(5) can be shown similarly as in [27]. We only sketch the proof since it is highly technical and does not involve new ideas. First, we rewrite (4)-(5), setting $\rho=\rho_\varepsilon$, $u=u_\varepsilon$, and $\varepsilon=1$. Taking the divergence of (5) and replacing $\operatorname{div}\partial_t(\rho u)$ by (4), which has been differentiated with respect to time, we obtain

$$\begin{aligned}
\partial^2_{tt}\rho - \frac{1}{\gamma}\Delta\rho^\gamma + 2\rho\sigma'(\rho)^2\Delta^2\rho &= -\operatorname{div}\operatorname{div}(\rho u\otimes u) - \operatorname{div}(\rho u^\perp) \\
&\quad + 2\operatorname{div}\operatorname{div}(\mu(\rho)D(u)) + F[\rho],
\end{aligned}$$

where $F[\rho] = 2\operatorname{div}(\rho\nabla(\sigma'(\rho)\Delta\sigma(\rho))) - 2\rho\sigma'(\rho)^2\Delta^2\rho$ involves only three derivatives. This formulation allows one to treat the momentum equation as a nonlinear fourth-order wave equation for which existence and regularity results can be applied. In order to derive some regularity for the velocity, Li and Marcati [27] assumed that $\operatorname{curl} u = 0$. Then u is reconstructed from the problem

$$\operatorname{div} v = -\frac{1}{\rho}(\partial_t\rho + \nabla\rho\cdot u), \quad \operatorname{curl} v = 0, \quad \int_{\mathbb{T}^2} v(t)dx = \bar{u}(t).$$

Theorem 2.1 in [27] gives the existence of a unique solution $u\in H^{s+1}(\mathbb{T}^2)$ to this problem, provided that the right-hand side satisfies $-(\partial_t\rho+\nabla\rho\cdot u)/\rho \in H^s(\mathbb{T}^2)$. Actually, Li and Marcati replace the right-hand side by $-(\partial_t\rho+\nabla\rho\cdot u)/\psi$, where ψ solves the mass equation

$$\partial_t\psi + \psi\operatorname{div} v + u\cdot\nabla\rho = 0, \quad t>0, \quad \psi(0)=\rho^0.$$

The reason is that this equation can be solved explicitly, yielding strictly positive solutions ψ. The existence proof is based on an iteration scheme: Given (ρ_p,ψ_p,u_p,v_p), solve

$$\operatorname{div} v_{p+1} = f_p(t), \quad \operatorname{curl} v_{p+1} = 0, \quad \int_{\mathbb{T}^2} v_{p+1}(t)dx = \bar{u}(t),$$

$$\partial_t \psi_{p+1} + \psi_{p+1} \operatorname{div} v_p + u_p \cdot \nabla \rho_p = 0, \quad t > 0, \quad \psi(0) = \rho^0,$$
$$\partial_{tt}^2 \rho_{p+1} - \frac{1}{\gamma} \Delta \rho_{p+1}^\gamma + \psi_p \sigma'(\psi_p)^2 \Delta^2 \rho_{p+1} = g_p(t), \quad t > 0,$$
$$\rho_{p+1}(0) = \rho^0, \ \partial_t \rho_{p+1}(0) = -\rho^0 \operatorname{div} u^0 - \nabla \rho^0 \cdot u^0,$$
$$\partial_t u_{p+1} + u_{p+1}^\perp = h_p(t),$$

where $f_p(t)$, $g_p(t)$, and $h_p(t)$ contain the remaining terms (see [27, Section 3] for details). The existence of solutions to these linear problems follows from ODE theory and the theory of wave equations. The main effort is now to derive uniform estimates in Sobolev spaces $H^k(\mathbb{T}^2)$. This is done by multiplying the above equations by suitable test functions and assuming that $T > 0$ is sufficiently small. By compactness, there exists a subsequence of $(\rho_p, \psi_p, u_p, v_p)$ which converges in a suitable Sobolev space to (ρ, ψ, u, v) as $p \to \infty$. This limit allows us also to show that $\rho = \psi \geq 0$ and $u = v$. This shows the existence of local smooth solutions under the assumption of irrotational flow $\operatorname{curl} u = 0$.

Next, we prove the existence of local smooth solutions to the quasi-geostrophic equation (7). We set $\mu := \mu(1) > 0$.

Theorem 2 (Local existence for the quasi-geostrophic equation). *Let $\phi_0 \in C^\infty(\mathbb{T}^2)$. Then there exists $T > 0$ and a smooth solution ϕ to* (7)-(8) *for $0 \leq t \leq T$.*

Proof. The idea of the proof is to apply the theory of linear semigroups. Let $p > 2$ and let $A_p : W^{2,p}(\mathbb{T}^2) \to \mathbb{R}$, $A_p(u) = -\mu \Delta u + u$. Then A_p is a sectorial operator satisfying $\Re(\lambda) = 1$ for all $\lambda \in \sigma(A_p)$, where $\sigma(A_p)$ denotes the spectrum of A_p. Consequently, A_p possesses the fractional powers A_p^β for $\beta \geq 0$, defined on the domain $X^{\beta,p} = D(A_p^\beta)$. This space, endowed with its graph norm, satisfies $X^{\beta,p} \hookrightarrow W^{k,q}(\mathbb{T}^2)$ if $k - 2/q < 2\beta - 2/p$, $q \geq p$ [21, Theorem 1.6.1]. Let $\max\{1 - 1/p, 1/2 + 1/(2p)\} < \beta < 1$ and set $X := X^{\beta,p}$. The operator A_p generates an analytical semigroup e^{-tA_p} $(t \geq 0)$ [21, Theorem 1.3.4], and the following estimates hold for all $t > 0$ [21, Theorem 1.4.3]:

$$\|A_p e^{-tA_p} u\|_{L^p(\mathbb{T}^2)} \leq C t^{-\beta} e^{-\delta t} \|u\|_{L^p(\mathbb{T}^2)},$$
$$\|(e^{-tA_p} - I)v\|_{L^p(\mathbb{T}^2)} \leq C t^\beta \|A_p v\|_{L^p(\mathbb{T}^2)} \leq C t^\beta \|v\|_X$$

for $0 < \delta < 1$, $u \in L^p(\mathbb{T}^2)$, and $v \in X$.

Next, we reformulate (7). Set $u = \phi - \Delta\phi$. Then (7) can be written as a system of two second-order equations:

$$-\Delta\phi + \phi = u \quad \text{in } \mathbb{T}^2, \ t > 0, \tag{22}$$
$$\partial_t u - \mu \Delta u + u = (\nabla^\perp \phi \cdot \nabla)(\phi - u) + \mu(u - \phi) + u. \tag{23}$$

We employ a fixed-point argument. Let $T > 0$ and $R > 0$. We introduce the spaces $Y = C^0([0,T]; X)$ and $B_R = \{u \in Y : \|u - u^0\|_Y \leq R\}$, where $u^0 = -\Delta\phi^0 + \phi^0 \in C^\infty(\mathbb{T}^2)$. Given $u \in Y \subset C^0([0,T]; L^p(\mathbb{T}^2))$, let $\phi \in L^\infty(0,T; W^{2,p}(\mathbb{T}^2))$ be the unique solution to (22) satisfying the elliptic estimate $\|\phi\|_{W^{2,p}(\mathbb{T}^2)} \leq C\|u\|_{L^p(\mathbb{T}^2)}$. Then define

$$J(u) = e^{-tA_p} u^0 + \int_0^t e^{(t-s)A_p} F(\phi(s), u(s)) ds, \quad \text{where}$$

$$F(\phi,u) = (\nabla^\perp\phi\cdot\nabla)(\phi-u) + \mu(u-\phi) + u.$$

Using the continuous embedding $W^{2,p}(\mathbb{T}^2) \hookrightarrow W^{1,2p}(\mathbb{T}^2)$ and the elliptic estimate for ϕ, we infer the estimate

$$\begin{aligned}\|F(\phi,u)\|_{L^\infty(0,T;L^p(\mathbb{T}^2))} &\le C\|u\|_{L^\infty(0,T;W^{1,2p}(\mathbb{T}^2))}\big(1+\|u\|_{L^\infty(0,T;W^{1,2p}(\mathbb{T}^2))}\big)\\ &\le C\|u\|_{L^\infty(0,T;X)}\big(1+\|u\|_{L^\infty(0,T;X)}\big)\\ &= C\|u\|_Y(1+\|u\|_Y).\end{aligned}$$

The last inequality follows from the embedding $X \hookrightarrow W^{1,2p}(\mathbb{T}^2)$ which holds for $\beta > 1/2 + 1/(2p)$.

We show that J maps B_R into B_R and that $J : B_R \to B_R$ is a contraction for sufficiently small $T > 0$. Let $T > 0$ be such that $\|(e^{-tA_p} - I)u_0\|_{L^p(\mathbb{T}^2)} \le CT^\beta\|u^0\|_X \le R/2$. Then, for $u \in B_R$,

$$\begin{aligned}\|J(u)-u^0\|_Y &\le \sup_{0<t<T}\|(e^{-tA_p}-I)u^0\|_{L^p(\mathbb{T}^2)}\\ &\quad + \sup_{0<t<T}\int_0^t \|A_pe^{-tA_p}F(\phi(s),u(s))\|_X ds\\ &\le \frac{R}{2} + \sup_{0<t<T}\int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\|F(\phi(s),u(s))\|_X ds\\ &\le \frac{R}{2} + \frac{CT^{1-\beta}}{1-\beta}\|u\|_Y(1+\|u\|_Y) \le R,\end{aligned}$$

if $T > 0$ is sufficiently small, using that $u \in B_R$. Thus $J(u) \in B_R$. In a similar way, we show that, for given u, $v \in B_R$,

$$\|J(u)-J(v)\|_Y \le \frac{CT^{1-\beta}}{1-\beta}(\|u\|_Y + \|v\|_Y)\|u-v\|_Y.$$

Again, choosing $T > 0$ small enough, J becomes a contraction, and the fixed-point theorem of Banach provides the existence and uniqueness of a mild solution on $[0,T]$.

It remains to prove that the mild solution is smooth. Since $\beta > 1 - 1/p$, we have $X \hookrightarrow W^{2,p/2}(\mathbb{T}^2)$ and hence $u \in L^\infty(0,T;W^{2,p/2}(\mathbb{T}^2)) \subset L^\infty(0,T;W^{1,p}(\mathbb{T}^d))$. Furthermore, $\nabla\phi \in L^\infty(0,T;W^{1,p}(\mathbb{T}^2)) \subset L^\infty(0,T;L^\infty(\mathbb{T}^2))$ (here, we use $p > 2$). This shows that $\partial_t u + A_p(u) \in L^\infty(0,T;L^p(\mathbb{T}^2))$. Parabolic theory implies that $u \in L^q(0,T;W^{2,p}(\mathbb{T}^2))$ for all $q < \infty$. This improves the regularity of ϕ to $\phi \in L^q(0,T;W^{4,p}(\mathbb{T}^2))$. Hence, $\partial_t u + A_p(u) \in L^q(0,T;L^\infty(\mathbb{T}^2))$, and a bootstrap procedure finishes the proof.

Acknowledgement. The first author acknowledges partial support from the Austrian Science Fund (FWF), grants P22108, P24304, and I395 and from the National Science Council of Taiwan (NSC) through the Tsungming Tu Award. The second author was supported by the NSC, grant NSC101-2115-M-009-008-MY2. The third author acknowledges partial support from the Tsz-Tza Foundation of the Institute of Mathematics, Academia Sinica (Taipei, Taiwan). He thanks Clément Mouhot for his kind invitation to visit the University of Cambridge during the academic years 2011-2013. Part of this work was written during the stay of the first author at the Center of Mathematical Modeling and Scientific Computing, National Chiao Tung University (Hsinchu, Taiwan), and at the Institute of Mathematics, Academia Sinica (Taipei, Taiwan); he thanks Chi-Kun Lin and Tai-Ping Liu for their kind hospitality.

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