Long-time behavior of the nonlinear Schrödinger–Langevin equation

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Abstract

We consider the large time behavior for nonlinear Schrödinger–Langevin equation in one dimension for WKB-initial data with different density at left/right far fields. We show that the momentum damping overwhelms the quantum dispersion. Thus, unlike those in scattering theory, the solution tends to an asymptotic state determined by a porous media equation. More precisely, the total density tends pointwise to a nonlinear diffusion wave and the phase tends to a corresponding function.

1 Introduction and Main results

The theory of quantum mechanics was employed to deal with the dissipative system which were observed, for example, in heavy ion physics and frictional phenomena in fission, etc [25, 20]. Recently, the nonlinear Schrödinger–Langevin equation is taken into granted to describe the dissipative process due to frictional force, for instance, in the motion of a Brownian particle in heat bath by Kostin [15], to characterize directly a class of nonlinear quantum mechanics through nonlinear gauge generalization by Doebner-Goldin-Nattermann [5], and to study the motion of charged (quantum) particles in semiconductor of nano-size [12, 19], and so on. The starting point for the derivation of Schrödinger–Langevin equation is the (quantum) Langevin equation. It is well-known that the Langevin equation has been widely used in order to investigate the diffusion of Brownian particles, dissipation and other non-equilibrium phenomena. In classical mechanics, the Langevin equation for a Brownian particle of mass m acted on by an external force $F(x)$ is

$$\begin{cases} \dot{x}(t) = \dfrac{1}{m}\mathbf{k}, \\ \dot{\mathbf{k}}(t) = -\dfrac{\xi}{m}\mathbf{k} + F(x) + \Gamma(t), \end{cases} \tag{1.1}$$

where $\mathbf{k} = m\dot{x}$ is the momentum, $\xi > 0$ is the friction constant, and $\Gamma(t)$ is the stochastic force due to heat bath. This force a purely random centered Gaussian process characterized by

$$\langle \Gamma(t) \rangle = 0, \quad \langle \Gamma(t), \Gamma(t') \rangle = 2\xi\kappa T\delta(t - t'),$$

where $T > 0$ is the temperature of heat bath and κ is the Boltzmann constant. Based on this fundamental equation (1.1) one can derive the well-known Fokker-Planck equations [23].

In quantum mechanical analogy, Ford-Kac-Mazur [7, 8] have proposed the quantum Langevin equation which is the Heisenberg equation of motion for the (operator) coordinate of a Brownian particle coupled to a heat bath:

$$\begin{aligned}\dot{X}(t) =& \frac{1}{m}\mathbb{K}, \\ \dot{\mathbb{K}}(t) =& -\frac{\xi}{m}\mathbb{K} + F(X) + \Gamma(t).\end{aligned} \tag{1.2}$$

Here $\mathbb{K}$ is the Heisenberg momentum operator, and X is the Heisenberg position operator. Starting with a friction term propositional to the expectation of the Heisenberg momentum operator $\mathbb{K}$ in the Ehrenfest equation (the second equation above), Kostin [15] was able to derived the nonlinear Schrödinger–Langevin equation for a Brownian particle interacting with a thermal background.

In general, the nonlinear Schrödinger–Langevin for the wave function Ψ takes the form

$$i\varepsilon\partial_t\Psi = -\frac{1}{2}\varepsilon^2\Delta\Psi + h(|\Psi|^2)\Psi + \frac{1}{\tau}S\Psi, \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R}_+, \tag{1.3}$$

$$S = \frac{1}{2}\varepsilon\ln(\Psi/\Psi^*), \tag{1.4}$$

where $d \geq 1$, $\varepsilon > 0$ is the scaled Planck constant, $\tau > 0$ is the scaled frictional constant, and Ψ^* denotes the complex conjugate of the wave function Ψ. The function $h(|\Psi|^2)$ represents the self-interaction potential. We shall assume $h' > 0$. Physically it means that the interaction of particles is repulsive. There are other derivations of the Schrödinger–Langevin equation based on different assumptions, see [14, 27, 4, 26, 9].

With the frictional force (1.4) acting up, the dynamics of the wave function Ψ of Eq. (1.3) is completely different from the classical one for nonlinear Schrödinger equation. In fact, it was proven that Schrödinger-Langevin equation ususally can have no solitary type solutions in the damped free-particle case in energy sapce [1], and that the coherent quantum-oscillation trajectories are damped due to the nonlinear friction force in the Shcrödinger-Langevin equation where the coherent oscillations decay exponentially with time [24].

We are interested in the mathematical analysis on the large time behavior of the macroscopic observable–the mass and the momentum of the nonlinear Schrödinger–Langevin equation caused by the nonlinear frictional effect. Roughly speaking, the new frictional term $S\Psi$ on the right hand side of (1.3) caused by the purely random force through Langevin equation is dissipative. Thus, we may expect a different asymptotic profile of the wave function in large time. To have an intuition, we apply Madelung's idea [18] to describe quantum systems in terms of a fluid-dynamical description of the macroscopic observables such as mass, momentum, and energy. We look for the solution of the WKB-form $\Psi = \sqrt{\rho}\exp(iS/\varepsilon)$ of Eq. (1.3)–(1.4), substitute it into equations, and separate the real part and image part respectively, we can obtain the Madelung fluid-type equations for the particle density ρ and the momentum $J = \rho\nabla S$ for irrotational flow

$$\partial_t\rho + \mathrm{div}(\rho\nabla S) = 0, \tag{1.5}$$

$$\partial_t(\rho\nabla S) + \mathrm{div}\,(\rho\nabla S \otimes \nabla S) + \nabla p(\rho) = \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{1}{\tau}\rho\nabla S, \tag{1.6}$$

where the pressure $p = p(\rho)$ satisfies $p'(\rho) = \rho h'(\rho)$, and the i-th component of the convective term $\mathrm{div}(\rho u \otimes u)$ equals $\sum_{k=1}^d \partial_{x_k}(\rho u_i u_k)$. Let us introduce the re-scaling:

$$t \to \tau t,\ \rho^\tau = \rho(\frac{t}{\tau}, x),\ S^\tau = \frac{1}{\tau}S(\frac{t}{\tau}, x), \tag{1.7}$$

to transform (1.5)–(1.6) into

$$\partial_t\rho + \mathrm{div}(\rho\nabla S) = 0, \tag{1.8}$$

$$\tau^2\partial_t(\rho\nabla S) + \tau^2\mathrm{div}\,(\rho\nabla S\otimes\nabla S) + \nabla p(\rho) = \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \rho\nabla S, \tag{1.9}$$

Performing the formal limits $\varepsilon \to 0$ and $\tau \to 0$, we obtain the following nonlinear parabolic equation for density

$$\partial_t\rho = \Delta p(\rho). \tag{1.10}$$

Thus, instead of convergence to that of free Schrödinger equation, we expect the density may tend to the self-similar solutions of the parabolic equation (1.10). In the present paper, we justify above expected long-time behavior for nonlinear Schrödinger–Langevin equation (1.3)–(1.4) in one-dimension for the following WKB initial data:

$$\Psi(x,t=0) = \Psi_0(x) = \sqrt{\rho_0(x)}e^{iS_0(x)/\varepsilon}, \tag{1.11}$$

$$\rho_0(\pm\infty) = \rho_\pm > 0, \quad S_0(\pm\infty) = -h(\rho_\pm). \tag{1.12}$$

As shown in [6], Eq. (1.10) in one-dimension admits a unique self-similar solution up to a position shift, the nonlinear diffusion wave. It has the form $\rho(x,t) = W(\xi)$, ($\xi = \frac{x}{\sqrt{1+t}}$) with the boundary conditions:

$$W(\pm\infty) = \rho_\pm. \tag{1.13}$$

Note that the mass ρ satisfies the conservation law (1.5). When the initial density ρ_0 is a perturbation of the nonlinear diffusion wave, it causes a shift of the nonlinear diffusion wave in the following sense[10]:

$$\int_{-\infty}^{\infty}[(\rho_0(x) - W(x+x_0,t=0)]dx = \tau(J_+ - J_-), \tag{1.14}$$

where the constant $x_0 \in \mathbb{R}$ is the shift, and $J_\pm = \rho_\pm u_\pm$. As it was shown in [10], the momentum (J_-, J_+) can be set to be zero at infinity. In fact, if not, due to the damping of the momentum equation (at infinity), we can define $J_e(x,t)$ and $\rho_e(x,t)$ as the follows:

$$J_e(x,t) = J_-e^{-\frac{1}{\tau}t} + (J_+ - J_-)e^{-\frac{1}{\tau}t}\int_{-\infty}^{\infty}\tilde{\rho}(x)dx, \tag{1.15}$$

$$\rho_e(x,t) = \frac{J_+ - J_-}{\tau}\tilde{\rho}(x)e^{-\frac{1}{\tau}t}, \tag{1.16}$$

where $\tilde{\rho}(x) \geq 0$ belongs to $C_0^\infty(\mathbb{R})$ and satisfies

$$\int_{-\infty}^{\infty}\tilde{\rho}(x)dx = 1.$$

The functions J_e carries the initial momentum at infinity, whereas ρ_e contains the mass induced by J_e at far fields. Then the shift x_0 is determined by

$$\int_{-\infty}^{\infty}[\rho_0(x) - W(x+x_0,t=0) - \rho_e(x,t=0)]dx = 0. \tag{1.17}$$

By removing J_e from J and ρ_e from ρ, we may assume

$$J_\pm = 0. \tag{1.18}$$

It is convenient to investigate the large time behavior of the IVP for NLS (1.3)–(1.4) and (1.11)–(1.12) in terms of the physical quantities, the amplitude $n = \sqrt{\rho}$ and the momentum $J = n^2S_x$. The macroscopic equations take

$$2nn_t + J_x = 0, \tag{1.19}$$

$$J_t + \left(\frac{J^2}{\rho} + P(n)\right)_x = \frac{1}{2}\varepsilon^2 n^2 \left(\frac{n_{xx}}{n}\right)_x - \frac{J}{\tau}, \tag{1.20}$$

where

$$P(n) = p(n^2). \tag{1.21}$$

The initial and boundary conditions are given by

$$n(x,0) = n_0(x) > 0, \quad J(x,0) = J_0(x), \tag{1.22}$$

$$n(\pm\infty, t) = n_\pm := \sqrt{\rho_\pm}, \quad J(\pm\infty,t) = J_\pm = 0. \tag{1.23}$$

Set

$$\begin{aligned} z_0(x) = &\int_{-\infty}^{x} (n_0^2(y) - W(y + x_0, 0))dy, \\ w_0(x) = &n_0(x) - \sqrt{W(x+x_0,0)}, \quad \eta_0(x) = J_0(x) + p(W(x+x_0,0))_x. \end{aligned}$$

The main theorem on the large time behavior of IVP (1.19)–(1.23) is

Theorem 1.1 *Let $p'(\rho) > 0$ for $\rho > 0$, and $|n_+ - n_-| \ll 1$. Assume that $z_0 \in L^2(\mathbb{R})$, $w_0 \in H^5(\mathbb{R})$, $\eta_0 \in H^4(\mathbb{R})$ with $\|z_0\|_{L^2(\mathbb{R})} + \|w_0\|_{H^5(\mathbb{R})} + \|\eta_0\|_{H^4(\mathbb{R})}$ sufficiently small, but independent of ε. Then, there is a global classical solution (n, J) of IVP (1.19)–(1.23) such that*

$$\|n(\cdot,t) - \sqrt{W(\cdot + x_0, t)}\|_{H^5} + \|J(\cdot,t) + \tau p\left(W(\cdot + x_0,t)\right)_x\|_{H^4} \to 0,$$

as $t \to \infty$. Moreover, it holds

$$\begin{aligned} &\|n(\cdot,t) - \sqrt{W(\cdot + x_0,t)}\|_{L^\infty} \le C(1+t)^{-3/4}, \\ &\|J(\cdot,t) + \tau p\left(W(\cdot + x_0,t)\right)_x\|_{L^\infty} \le C(1+t)^{-5/4}. \end{aligned}$$

¿From the solution (ρ, u) of IVP (1.19)–(1.23), we can construct the solution of IVP for NLS (1.3)–(1.4) and (1.11)–(1.12). In fact, from (1.20), the equation for velocity $u = S_x$ is

$$u_t + \frac{1}{2}(u^2)_x + h(n^2)_x = \frac{1}{2}\varepsilon^2 \left(\frac{n_{xx}}{n}\right)_x - \frac{1}{\tau}u, \tag{1.24}$$

from which we reckon the total velocity satisfies

$$\int_{-\infty}^{\infty} u(x,t)dx = e^{-t/\tau}\int_{-\infty}^{\infty} u_0(x)dx - \tau[h(n_+^2) - h(n_-^2)](1 - e^{-t/\tau}) < \infty.$$

Thus, the wave function $\Psi(x,t)$

$$\Psi(x,t) = n(x,t)e^{iS(x,t)/\varepsilon}$$

with

$$S(x,t) = -\tau h(W_-) + \int_{-\infty}^{x} u(y,t)dx \tag{1.25}$$

is well-defined and satisfies IVP (1.3)–(1.4) and (1.11)–(1.12).

Set

$$\phi_0 = S_0(x) + \tau h\left(W(x + x_0, t = 0)\right). \tag{1.26}$$

The large time behavior for the NLS (1.3)–(1.4) and (1.11)–(1.12) is then obtained as the follows:

Theorem 1.2 *Let $h'(\rho) > 0$ for $\rho > 0$, and $|n_+ - n_-| \ll 1$. Assume that $(z_0, \phi_0) \in L^2(\mathbb{R})$, $w_0 \in H^5(\mathbb{R})$, $\eta_0 \in H^4(\mathbb{R})$ with $\|z_0\|_{L^2(\mathbb{R})} + \|w_0\|_{H^5(\mathbb{R})} + \|\eta_0\|_{H^4(\mathbb{R})}$ sufficiently small. Then, there is a global classical solution $\Psi = ne^{iS/\varepsilon}$ of IVP (1.3)–(1.4) and (1.11)–(1.12) such that*

$$\|(\Psi - \tilde{\Psi})(\cdot, t)\|_{H^4(\mathbb{R})} \to 0, \quad as \quad t \to \infty, \tag{1.27}$$

where $\tilde{\Psi} = \sqrt{W(\xi)}e^{-i\tau h(W(\xi))/\varepsilon}$, $\xi = (x + x_0)/\sqrt{1+t}$. Moreover, it holds

$$\|(n(\cdot, t) - \sqrt{W(\cdot + x_0, t)},\ S(\cdot, t) + \tau h\left(W(\cdot + x_0, t)\right))\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-3/4}. \tag{1.28}$$

2 Nonlinear diffusion waves

We list some known results concerning the self-similar solution of the nonlinear parabolic equation (1.10) in this section.

Assume that the pressure-density functions satisfy $p'(\rho) > 0$ and τ, ε are set to be one. Then the nonlinear parabolic equation (1.10) reads:

$$\rho_t = p(\rho)_{xx}, \qquad p'(\rho) > 0, \tag{2.1}$$

which possesses a unique self-similar solution $w(x,t)$ (see [6])

$$\rho(x,t) \triangleq W(\zeta), \quad \zeta = \frac{x}{\sqrt{t+1}},$$

satisfying

$$W''(\zeta) + \frac{p''(W(\zeta))W'(\zeta) - \frac{1}{2}\zeta}{p'(W(\zeta))}W'(\zeta) = 0,$$
$$W(\pm\infty) = \rho_\pm, \quad (\rho_+, \rho_- > 0).$$

This solution is increasing if $\rho_- < \rho_+$ and decreasing if $\rho_- > \rho_+$, and satisfies

$$\sum_{k=1}^{6} |\frac{d^k}{d\zeta^k}\Phi(\zeta)| + |W(\zeta) - \rho_+|_{\zeta>0} + |W(\zeta) - \rho_-|_{\zeta<0} \le C\delta e^{-c\zeta^2},$$
$$|W_t(x,t)| \le C\delta(1+t)^{-1},\ |W_x(x,t)| \le C\delta(1+t)^{-\frac{1}{2}},$$

where and throughout $\delta = |\rho_+ - \rho_-|$.

We introduce a new variable

$$\tilde{n}(x,t) = \sqrt{W(x + x_0, t)}.$$

¿From (2.1), $\tilde{n}$ satisfies

$$\tilde{n}_t = \frac{1}{2\tilde{n}}p(\tilde{n}^2)_{xx}.$$

We have the following L^p-estimates of the derivatives of W and $\tilde{n}$ as ([17]):

Lemma 2.1 *Let W be the self-similar solution of (1.10) and (1.13) and let $\tilde{n} = \sqrt{W}$. Then it holds that*

$$\|\partial_t^k \partial_x^j W(.,t)\|_{L_p} \le C\delta(1+t)^{-k-\frac{j}{2}+\frac{1}{2p}}, \tag{2.2}$$

$$\|\partial_t^k \partial_x^j \tilde{n}(.,t)\|_{L_p} \le C\delta(1+t)^{-k-\frac{j}{2}+\frac{1}{2p}}, \tag{2.3}$$

for $k, j \ge 0$ and $p \in [1, \infty]$, where $C > 0$ is some constant.

In the following section, we will often use the Moser-type calculus inequalities [13]:

Lemma 2.2 *Let $f, g \in L^\infty \cap H^s$. Then, it holds*

$$\|\partial_x^\alpha (fg)\| \le C\|g\|_{L^\infty} \|\partial_x^\alpha f\| + C\|f\|_{L^\infty} \|\partial_x^\alpha g\|, \tag{2.4}$$

$$\|\partial_x^\alpha (fg) - f\partial_x^\alpha g\| \le C\|g\|_{L^\infty} \|\partial_x^\alpha f\| + C\|f\|_{L^\infty} \|\partial_x^{\alpha-1} g\|, \tag{2.5}$$

for $1 \le \alpha \le s$. Here, $\|\cdot\|$ denotes for L^2 norm.

3 The perturbed equations

To obtain energy and decay estimates, we shall work on two sets of perturbed equations. One is an equation for the integral of the perturbed mass and the perturbed momentum:

$$z(x,t) = \int_{-\infty}^{x} (\rho(x,t) - W(y + x_0, t))dy, \quad \eta = J + p(W)_x. \tag{3.1}$$

The other is the equation for the perturbed amplitude and the perturbed momentum:

$$w = n - \tilde{n}, \quad \eta = J + P(\tilde{n})_x. \tag{3.2}$$

We derive them and explain why we need to use both equations for energy estimates at the end of this section.

¿From (1.19)–(1.23), the corresponding IVP for (z, η) becomes

$$z_t + \eta = 0, \tag{3.3}$$

$$\eta_t + \left[\frac{(\eta - p(W)_x)^2}{W + z_x} + p(W + z_x) - p(W) \right]_x = \frac{1}{2}\varepsilon^2 (W + z_x) \left(\frac{(\sqrt{W + z_x})_{xx}}{\sqrt{W + z_x}} \right)_x - \eta + p(W)_{xt}, \tag{3.4}$$

$$z(x,0) = z_0(x), \ \eta(x,0) = \eta_0(x), \quad x \in \mathbb{R}. \tag{3.5}$$

¿From (3.3)–(3.5), follows the IVP for the damped "wave equation" for z

$$z_{tt} + z_t - (p'(W)z_x)_x + \frac{1}{4}\varepsilon^2 z_{xxxx} = (f_1 + f_2 + f_3)_x \,. \tag{3.6}$$

The corresponding initial data are

$$z(x,0) = z_0(x), \quad z_t(x,0) = -\eta_0(x). \tag{3.7}$$

Here,

$$f_1 = \frac{1}{4}\varepsilon^2 \frac{(W_x + z_{xx})^2}{W + z_x} - p(W)_t - \frac{1}{4}\varepsilon^2 W_{xx}, \tag{3.8}$$

$$f_2 = \frac{J^2}{\rho} = \frac{(p(W)_x + z_t)^2}{W + z_x}, \tag{3.9}$$

$$f_3 = p(W + z_x) - p(W) - p'(W)z_x. \tag{3.10}$$

and we have used

$$\rho \left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}} \right)_x = \frac{1}{2}\rho_{xxx} - \frac{1}{2} \left(\frac{\rho_x^2}{\rho} \right)_x .$$

We recall that we have assumed $p' > 0$. The term $(p'(W)z_x)_x$ is a diffusion term. We denote

$$\min p'(W) = \nu > 0. \tag{3.11}$$

¿From (1.19)–(1.20), we derive the "wave equation" for $n = \sqrt{\rho}$ as

$$n_{tt} + n_t + \frac{1}{n}n_t^2 - \frac{1}{2n}\left[P(n) + \frac{J^2}{n^2}\right]_{xx} + \frac{1}{4}\varepsilon^2 n_{xxxx} - \frac{1}{4}\varepsilon^2\frac{n_{xx}^2}{n} = 0,$$

where we recall $P(n) = p(n^2)$, and we have used the relation

$$\left[n^2\left(\frac{n_{xx}}{n}\right)_x\right]_x = n\left[n_{xxxx} - \frac{n_{xx}^2}{n}\right].$$

Recalling $w = n - \tilde{n}$ and $\eta = J + P(\tilde{n})_x$, then we obtain the equations for (w, η) as

$$2(\tilde{n} + w)w_t + 2\tilde{n}_t w + \eta_x = 0, \tag{3.12}$$

$$w_{tt} + w_t - (p'(W)w_x)_x + \frac{1}{4}\varepsilon^2 w_{xxxx} = g_1 + g_2 + g_3, \tag{3.13}$$

imposed with the initial values

$$\eta(x, 0) = \eta_0(x), \tag{3.14}$$

$$w(x, 0) = w_0 = n_0 - \tilde{n}, \quad w_t(x, 0) = \dot{w}_0(x) =: -\frac{\eta_{0x} + 2\tilde{n}w_0}{2(\tilde{n} + w_0)}. \tag{3.15}$$

Here,

$$g_1(x, t) = \frac{(\tilde{n}_t + w_t)^2}{\tilde{n} + w} + \frac{\varepsilon^2}{4}\frac{(\tilde{n}_{xx} + w_{xx})^2}{\tilde{n} + w} - \frac{\varepsilon^2}{4}\tilde{n}_{xxxx} - \tilde{n}_{tt}, \tag{3.16}$$

$$\begin{aligned} g_2(x, t) =& \frac{1}{2\sqrt{\rho}}\left[\frac{J^2}{\rho}\right]_{xx} \\ =& \frac{1}{2(\tilde{n} + w)}\left[\frac{(P(\tilde{n})_x - \eta)^2}{(\tilde{n} + w)^2}\right]_{xx} \end{aligned} \tag{3.17}$$

$$\begin{aligned} g_3(x, t) &= \left[p'((\tilde{n} + w)^2)(\tilde{n}_x + w_x)\right]_x - \left[p'(\tilde{n}^2)\tilde{n}_x\right]_x - \left[p'(\tilde{n}^2)w_x\right]_x \\ &= \left[(p'((\tilde{n} + w)^2) - p'(\tilde{n}^2))(\tilde{n}_x + w_x)\right]_x \end{aligned} \tag{3.18}$$

with η_x defined by (3.12).

There is a relation equation between w and z_x:

$$(2\tilde{n} + w)w = z_x \quad \text{or} \quad w = \frac{1}{2\tilde{n} + w}z_x, \tag{3.19}$$

which follows from $\rho = n^2$.

Below we shall use both equations for the energy estimates. Roughly speaking, the left-hand sides of both z-equation (3.6) and w-equation (3.13) produce two good terms in the energy estimates: the dissipation energies

$$\int_0^t \|z_x(s)\|^2\, ds, \quad \int_0^t \|w_x(s)\|^2\, ds$$

and the damping energies

$$\int_0^t \|z_t(s)\|^2\,ds, \quad \int_0^t \|w_t(s)\|^2\,ds.$$

The right-hand side of the w-equation (3.13) produces a term $\int_0^t \|w\|^2\,ds$ which cannot be controlled in the energy estimate for the w-equation, but it can be controlled by the dissipation energy of the z-equation, because $\|w\| \sim \|z_x\|$. On the other hand, the bad term on the right-hand side of the z-equation (3.6) is z_{xxx}, which produces $\int_0^t \left(\|z_{xt}\|^2 + \|z_{xx}\|^2\right) ds$ in the energy estimate. Thus, the energy estimate cannot be closed by itself. Fortunately, this term is bounded by $\int_0^t \left(\|w_t\|^2 + \|w_x\|^2 + \|w\|^2\right) ds$ from (3.19) and it can be controlled by the dissipation and damping energies of the w-equation and the z-equation. Notice that the term $\int_0^t \|z(s)\|^2\,ds$ does not appear in the energy estimate for the z-equation because its right-hand side is a derivative. Thus, the combination of the energy estimates for z and w can close both energy estimates.

4 A priori estimates

4.1 A priori assumption

In order to perform the a priori energy estimate, let us assume that it holds for local in time solutions that for $T \geq 0$,

$$\delta_T = \max_{0\leq t\leq T} \sum_{k=0}^{1} \left(\|\partial_t^k z(t)\| + \|\partial_t^k w(t)\|_{H^{5-2k}}\right) \ll 1 \tag{4.1}$$

Here, $\|\cdot\|_{H^s}$ is the Sobolev norm and $\|\cdot\|$ is the L^2 norm. Under the smallness assumption of $\delta_T + \delta$, our goal is to show that δ_T is bounded by $\delta_0 + \delta$, where

$$\delta_0 = \sum_{k=0}^{1} \left(\|\partial_t^k z(0)\| + \|\partial_t^k w(0)\|_{H^{5-2k}}\right) \tag{4.2}$$

involves only the initial data.

Lemma 4.1 *Under the assumption (4.1), we have*

$$\frac{1}{2}\sqrt{\rho_-} \leq \tilde{n} + w \leq \frac{3}{2}\sqrt{\rho_+}, \quad \frac{1}{2}\rho_- \leq \rho \leq \frac{3}{2}\rho_+, \tag{4.3}$$

Proof: ¿From (3.19), $\|z_x\| \leq O(\|w\|)$. The lemma follows easily from the smallness of $\|z\|_{H^1}$, $\|w\|_{H^1}$ and Sobolev embedding. □

We have the following relations between z_x and w.

Lemma 4.2 *It holds that*

$$\|w\| \sim \|z_x\| \tag{4.4}$$

and

$$\|\partial_t^k \partial_x^j z_x\| = \sum_{l=0}^{k} \sum_{i=0}^{j} O(\delta_T + \delta)\|\partial_t^l \partial_x^i w\|, \quad 0 \leq 2k + j \leq 5, \tag{4.5}$$

provided $\delta_T \ll 1$.

Proof: The proof follows easily from the relation:

$$z_x = (2\tilde{n} + w)w,$$

assumption (4.1), (2.3) and Lemma 4.1. □

We have some basic estimates.

Lemma 4.3 *Under the assumption (4.1), it holds that for* $0 \le t \le T$

$$\sum_{k=0}^{2} \|\partial_t^k w(t)\|_{H^{5-2k}}, \sum_{k=0}^{2} \|\partial_t^k z(t)\|_{H^{6-2k}},$$
$$\sum_{k=0}^{1} \|\partial_t^k \eta(t)\|_{H^{4-2k}}, \sum_{k=0}^{1} \|\partial_t^k J(t)\|_{H^{4-2k}} = O(\delta_T + \delta). \tag{4.6}$$

Proof: ¿From w-equation (3.13), w_{tt} can be expressed in terms of $\partial_t^k \partial_x^{4-2k} w$ with $k \le 1$. ¿From assumption (4.1), we get $\|w_{tt}\|_{H^1} = O(\delta_T)$. The estimates for $\|z_x\|_{H^5}, \|z_{xt}\|_{H^3}$ follows from Lemma 4.2 and assumption (4.1). The estimate for $\|z_{tt}\|_{H^2}$ follows from the z-equation (3.6). The estimates for η comes from (3.3), (3.12), (4.1). ¿From $J = \eta - P(\tilde{n})_x$ and the estimates of η and Lemma 4.2, we get the estimates for J. □

¿From the relations: $\eta = -z_t$, $z_x = (2\tilde{n} + w)w$, $2(\tilde{n} + w)w_t + 2\tilde{n}_t w = -\eta_x$, we can get the following equivalent relations.

Lemma 4.4 *Under the assumption (4.1), the following norms are equivalent whenever one of them is small:*

$$\|z\|_{H^6} + \|z_t\|_{H^4} \sim \|z\|_{H^6} + \|\eta\|_{H^4}$$
$$\sim \|z\| + \|w\|_{H^5} + \|\eta\|_{H^4} \sim \|z\| + \|w\|_{H^5} + \|z_t\| + \|w_t\|_{H^3}. \tag{4.7}$$

We recall that the nonlinear terms have the following expression:

$$f_1 = \frac{\varepsilon^2}{4} \frac{(W_x + z_{xx})^2}{W + z_x} - p(W)_t - \frac{\varepsilon^2}{4} W_{xx},$$
$$f_2 = \frac{(p(W)_x + z_t)^2}{W + z_x},$$
$$f_3 = p(W + z_x) - p(W) - p'(W) z_x.$$

and

$$g_1 = \frac{(\tilde{n}_t + w_t)^2}{\tilde{n} + w} + \frac{\varepsilon^2}{4} \frac{(\tilde{n}_{xx} + w_{xx})^2}{\tilde{n} + w} - \frac{\varepsilon^2}{4} \tilde{n}_{xxxx} - \tilde{n}_{tt},$$
$$g_2 = \frac{1}{2(\tilde{n} + w)} \left[\frac{(P(\tilde{n})_x - \eta)^2}{(\tilde{n} + w)^2} \right]_{xx}$$
$$g_3 = \left[(p'((\tilde{n} + w)^2) - p'(\tilde{n}^2))(\tilde{n}_x + w_x) \right]_x.$$

¿From Lemmas 2.1, 4.1, 4.3, we have the following a priori estimates.

Lemma 4.5 *Under the assumption (4.1), the nonlinear terms have the following a-priori estimates:*

$$f_1 = O(\delta_T + \delta) z_{xx} + O(\delta) r_2,$$
$$f_2 = O(\delta_T + \delta) z_t + O(\delta) r_2,$$
$$f_3 = O(\delta_T) z_x,$$

where the function $r_k(x,t)$ *is related to the* k^{th} $x-$*derivative of* W. *It is defined such that*

$$\|r_k(\cdot, t)\|_{L^p} \le C(1 + t)^{-k/2 + 1/2p} \quad k = 0, 1, 2, \cdots. \tag{4.8}$$

Lemma 4.6 *Under the assumption (4.1), the nonlinear terms have the following a-priori estimates:*

$$\begin{aligned} g_1 &= (a_1 w_x)_x + O(\delta+\delta_T)w_t + O(\delta)r_4 \\ g_2 &= (a_2 w_x)_x + (b_2 w_t)_x + O(\delta+\delta_T)(w+w_x+w_t) + O(\delta)r_4 \\ g_3 &= (a_3 w_x)_x + (\delta_T+\delta)(w+w_x) \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{\epsilon^2}{4}\frac{2\tilde{n}_{xx}+w_{xx}}{\tilde{n}+w}, \qquad a_2 = -\frac{J^2}{\rho^2}, \qquad b_2 = \frac{2J}{\rho}, \\ a_3 &= p'((\tilde{n}+w)^2) - p'(\tilde{n}^2) = O(w). \end{aligned}$$

Lemma 4.7 *Under the assumption (4.1), the higher order derivatives of the nonlinear terms have the following a-priori estimates: for* $0 \le j \le 3$,

$$\begin{aligned} \partial_x^j g_1 =& (a_1 \partial_x^{j+1} w)_x + O(\delta_T+\delta)\left(\sum_{i=1}^{j+1}\partial_x^i w + \sum_{i=1}^{j}\partial_x^i w_t\right) + O(\delta) r_{j+4}, \\ \partial_x^j g_2 =& (a_2 \partial_x^{j+1} w)_x + (b_2 \partial_x^j w_t)_x + O(\delta_T+\delta)\left(\sum_{i=1}^{j+1}\partial_x^i w + \sum_{i=1}^{j}\partial_x^i w_t\right) + O(\delta) r_{j+4} \\ \partial_x^j g_3 =& (a_3 \partial_x^{j+1} w)_x + O(\delta_T+\delta)\sum_{i=1}^{j+1}\partial_x^i w. \end{aligned}$$

4.2 Estimates for (z, z_t)

Lemma 4.8 *For the local in time solutions $z(t)$, it holds for $0 \le t \le T$ that*

$$\begin{aligned} &\frac{1}{2}\|z_t(t)\|^2 + \frac{1}{4}\|z(t)\|^2 + \int_{-\infty}^{\infty} p'(W) z_x^2\,dx + \frac{\varepsilon^2}{4}\|z_{xx}(t)\|^2 \\ &+ \int_0^t \left(\frac{\varepsilon^2}{4}\|z_{xx}\|^2 + \frac{1}{2}\|z_t\|^2 + \frac{1}{2}\int_{-\infty}^{\infty} p'(W) z_x^2 dx\right) ds \\ &\le O(\delta_0+\delta)^2 + (\alpha + O(\delta_T+\delta))\int_0^t \left(\|z_{xx}(s)\|^2 + \|z_{xt}(s)\|^2\right) ds, \end{aligned} \tag{4.9}$$

where α is a constant such that

$$\alpha + O(\delta+\delta_T) \le \frac{1}{10}\min(1,\nu). \tag{4.10}$$

Proof: Multiplying (3.6) with $(z+2z_t)$ and integrating over $\mathbb{R}$, we get after integration by parts

$$\begin{aligned} &\frac{d}{dt}\left(\|z_t\|^2 + \frac{1}{2}\|z\|^2 + \frac{\varepsilon^2}{4}\|z_{xx}\|^2 + \int_{-\infty}^{\infty} p'(W) z_x^2\,dx + \int_{-\infty}^{\infty} z_t z\,dx\right) \\ &+ \frac{\varepsilon^2}{4}\|z_{xx}\|^2 + \|z_t\|^2 + \int_{-\infty}^{\infty} p'(W) z_x \cdot (z+2z_t)_x\,dx = -\int_{-\infty}^{\infty}(z+2z_t)_x(f_1+f_2+f_3)dx. \end{aligned}$$

The diffusion term on the left-hand side has the following estimate:

$$\int_{-\infty}^{\infty} p'(W) z_x \cdot (z+2z_t)_x\,dx \ge \int_{-\infty}^{\infty}(p'(W) - O(\delta+\delta_T)) z_x^2\,dx + \frac{d}{dt}\int_{-\infty}^{\infty} p'(W) z_x^2\,dx$$

Using Lemma 4.5 for f_m and Cauchy's inequality, we get

$$\left|\int_{-\infty}^{\infty}(f_1+f_2+f_3)\cdot(z+2z_t)_x\,dx\right| \le \alpha(\|z_x\|^2+\|z_{xt}\|^2)+O(1)(\|f_1\|^2+\|f_2\|^2+\|f_3\|^2)$$
$$\le \alpha(\|z_x\|^2+\|z_{xt}\|^2)+O(\delta_T+\delta)(\|z_{xx}\|^2+\|z_x\|^2+\|z_t\|^2)+O(\delta^2)(1+t)^{-3/2}.$$

Combining these estimates, we get

$$\frac{d}{dt}\left(\|z_t\|^2+\frac{1}{2}\|z\|^2+\frac{\varepsilon^2}{4}\|z_{xx}\|^2+\int_{-\infty}^{\infty}p'(W)z_x^2\,dx+\int_{-\infty}^{\infty}z_t z\,dx\right)$$
$$+\frac{\varepsilon^2}{4}\|z_{xx}\|^2+(1-\alpha-O(\delta+\delta_T))\|z_t\|^2+\int_{-\infty}^{\infty}(p'(W)-\alpha-O(\delta+\delta_T))z_x^2\,dx$$
$$\le(\alpha+O(\delta_T+\delta))\left(\|z_{xx}\|^2+\|z_{xt}\|^2\right)+O(\delta^2)(1+t)^{-3/2}.$$

Integrating this in time from 0 to t, applying Cauchy's inequality for $\int z_t z\,dx$, we get (4.9), provided α and $\delta+\delta_T$ satisfy (4.10). □

4.3 Estimates for (w, w_t, w_{tt})

4.3.1 Basic estimates

Lemma 4.9 *For the local in time solutions w, it holds*

$$\frac{1}{2}\|w_t(t)\|^2+\frac{1}{4}\|w(t)\|^2+\frac{1}{2}\int_{-\infty}^{\infty}p'(W)w_x^2\,dx+\frac{\varepsilon^2}{4}\|w_{xx}(t)\|^2$$
$$+\int_0^t\left(\frac{\varepsilon^2}{4}\|w_{xx}(s)\|^2+\frac{1}{2}\|w_t(s)\|^2+\frac{1}{2}\int_{-\infty}^{\infty}p'(W)w_x^2dx\right)ds$$
$$\le O(\delta_0+\delta)^2+(\alpha+O(\delta_T+\delta))\int_0^t\|w(s)\|^2\,ds, \tag{4.11}$$

for $0\le t\le T$, provided that $\delta_T+\delta$ is small enough. Here, α is defined by (4.10).

Proof: Multiply (3.13) with $(w+2w_t)$ and integrate it by part over $\mathbb{R}$:

$$\frac{d}{dt}\left(\|w_t\|^2+\frac{1}{2}\|w\|^2+\frac{\varepsilon^2}{4}\|w_{xx}\|^2+\int_{-\infty}^{\infty}p'(W)w_x^2\,dx+\int_{-\infty}^{\infty}w_t w\,dx\right)$$
$$+\frac{\varepsilon^2}{4}\|w_{xx}\|^2+\|w_t\|^2+\int_{-\infty}^{\infty}p'(W)w_x^2\,dx$$
$$=\int_{-\infty}^{\infty}\partial_t(p'(W))w_x^2\,dx+\int_{-\infty}^{\infty}(w+2w_t)(g_1+g_2+g_3)dx=I_0+I_1+I_2+I_3.$$

¿From Lemma 2.1, the term I_0 has the following estimate:

$$I_0=\int_{-\infty}^{\infty}p'(W)_t w_x^2\,dx=O(\delta)\|w_x\|^2. \tag{4.12}$$

¿From Lemma 4.6, integration-by-part and Cauchy's inequality, we get

$$\int_{-\infty}^{\infty}wg_1dx\le-\int_{-\infty}^{\infty}a_1w_x^2\,dx-\int_{-\infty}^{\infty}(\partial_x a_1)w_x w\,dx$$

$$
\begin{aligned}
&+ C(\delta+\delta_T)\|w_t\|^2 + \alpha\|w\|^2 + C\delta^2(1+t)^{-7/2}\\
&\le C(\delta+\delta_T)(\|w_t\|^2 + \|w_x\|^2 + \|w\|^2) + \alpha\|w\|^2 + C\delta^2(1+t)^{-7/2},
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} 2w_t g_1 dx &\le -\int_{-\infty}^{\infty} a_1 2w_x w_{xt}\, dx\\
&\quad + C(\delta+\delta_T)\|w_t\|^2 + \alpha\|w_t\|^2 + C\delta^2(1+t)^{-7/2}\\
&\le -\frac{d}{dt}\int_{-\infty}^{\infty} a_1 w_x^2\, dx + O(\delta+\delta_T)(\|w_t\|^2 + \|w_x\|^2) + \alpha\|w_t\|^2 + C\delta^2(1+t)^{-7/2}.
\end{aligned}
$$

Here, we have used

$$
\|\partial_x a_1\|_{L^\infty}, \|\partial_t a_1\|_{L^\infty} = O(\delta+\delta_T),
$$

$$
\left|\int_{-\infty}^{\infty} \delta r_4 \cdot (w+w_t)\, dx\right| \le \alpha(\|w\|^2 + \|w_t\|^2) + O(\delta^2)(1+t)^{-7/2}.
$$

For g_2, we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} w g_2\, dx &\le \int_{-\infty}^{\infty} w\cdot [(a_2 w_x)_x + (b_2 w_t)_x]\, dx\\
&+ O(\delta_T+\delta)(\|w\|^2 + \|w_x\|^2 + \|w_t\|^2) + \alpha\|w\|^2 + C\delta^2(1+t)^{-7/2}\\
&\le O(\delta_T+\delta)(\|w_x\|^2 + \|w\|^2 + \|w_t\|^2) + \alpha\|w\|^2 + C\delta^2(1+t)^{-7/2},
\end{aligned}
$$

$$
\begin{aligned}
2\int_{-\infty}^{\infty} w_t g_2 dx &\le \int_{-\infty}^{\infty} 2w_t \cdot [(a_2 w_x)_x + (b_2 w_t)_x]\, dx\\
&+ O(\delta_T+\delta)(\|w\|^2 + \|w_x\|^2 + \|w_t\|^2) + \alpha\|w_t\|^2 + C\delta^2(1+t)^{-7/2}\\
&\le -\frac{d}{dt}\left(\int_{-\infty}^{\infty} a_2 w_x^2 dx\right) + O(\delta_T+\delta)(\|w\|^2 + \|w_x\|^2 + \|w_t\|^2)\\
&\quad + \alpha\|w_t\|^2 + C\delta^2(1+t)^{-7/2}.
\end{aligned}
$$

Here, we have used Lemma 4.3 and the estimates

$$
\|\partial_x a_2\|_{L^\infty}, \|\partial_t a_2\|_{L^\infty}, \|\partial_x b_2\|_{L^\infty}, = O(\delta+\delta_T),
$$

which also follow from Lemma 4.3. For g_3, we get

$$
\int_{-\infty}^{\infty} (w+2w_t) g_3\, dx \le -\frac{d}{dt}\int_{-\infty}^{\infty} a_3 w_x^2\, dx + O(\delta_T+\delta)(\|w_x\|^2 + \|w\|^2 + \|w_t\|^2).
$$

Here, we have used

$$
\|\partial_x a_3\|_{L^\infty}, \|\partial_t a_3\|_{L^\infty} = O(\delta+\delta_T).
$$

We combine the above estimates to get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|w_t\|^2 + \frac{1}{2}\|w\|^2 + \frac{\varepsilon^2}{4}\|w_{xx}\|^2 + \int_{-\infty}^{\infty} (p'(W) + a_1 + a_2 + a_3)\, w_x^2\, dx + \int_{-\infty}^{\infty} w_t w\, dx\right)\\
&\quad + \frac{\varepsilon^2}{4}\|w_{xx}\|^2 + (1 - 2\alpha - O(\delta+\delta_T))\|w_t\|^2 + \int_{-\infty}^{\infty} (p'(W) - O(\delta+\delta_T))\, w_x^2\, dx\\
&\le (\alpha + O(\delta+\delta_T))\|w\|^2
\end{aligned}
$$

Integrating it in time from 0 to t, then applying Cauchy's inequality for $\int ww_t$, using $p'(W) \geq \nu > 0$, and choosing α and $\delta + \delta_T$ to satisfy (4.10), we can obtain (4.11). $\square$

Proposition 4.10 *For local in time classical solution, it holds that*

$$\begin{aligned}&\|z(t)\|_{H^2}^2 + \|z_t(t)\|^2 + \|w(t)\|_{H^2}^2 + \|w_t(t)\|^2 \\ &+ \int_0^t \left(\|z_t(s)\|^2 + \|w(s)\|_{H^2}^2 + \|w_t(s)\|^2\right) ds \leq C(\delta_0 + \delta)^2.\end{aligned} \tag{4.13}$$

provided $\delta_T + \delta$ is small enough.

Proof: We add (4.9) and (4.11) together. The terms on its right-hand side are $\int_0^t (\|z_{xx}(s)\|^2 + \|z_{xt}(s)\|^2)\, ds$ and $\int_0^t \|w(s)\|^2\, ds$, which can be estimated through the relations in Lemma 4.2 as the follows:

$$\int_0^t \|w(s)\|^2 \leq \int_0^t O(1)\|z_x(s)\|^2,$$

and

$$\int_0^t \|z_{xt}\|^2 \leq \int_0^t O(\delta + \delta_T)^2(\|w\|^2 + \|w_t\|^2) \leq \int_0^t O(\delta + \delta_T)^2(\|z_x\|^2 + \|w_t\|^2)$$

$$\int_0^t \|z_{xx}\|^2 \leq \int_0^t O(\delta + \delta_T)^2(\|w\|^2 + \|w_x\|^2) \leq \int_0^t O(\delta + \delta_T)^2(\|z_x\|^2 + \|w_x\|^2)$$

These terms can be absorbed into the damping and diffusion terms of z and w on the left-hand side, provided $\delta + \delta_T$ is sufficiently small. $\square$

4.3.2 Higher order estimates

Applying the similar procedure in proving Lemma 4.9, we further estimate higher order derivatives of w as the follows. We perform $\int_{-\infty}^{\infty} \partial_x^j(3.13) \cdot \partial_x^j(w + 2w_t)\, dx$. After integrating by part, we get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|\partial_x^j w\|^2 + \|\partial_x^j w_t\|^2 + \frac{\varepsilon^2}{4}\|\partial_x^{j+2} w\|^2 + \int_{-\infty}^{\infty}\left(\partial_x^j w_t \cdot \partial_x^j w + p'(W)|\partial_x^{j+1} w|^2\right) dx\right] \\ &+ \int_{-\infty}^{\infty} p'(W)|\partial_x^{j+1} w|^2 + \|\partial_x^j w_t\|^2 + \frac{\varepsilon^2}{4}\|\partial_x^{j+2} w\|^2 \\ &= I_0 + \int_{-\infty}^{\infty} (\partial_x^j w + 2\partial_x^j w_t)\partial_x^j(g_1 + g_2 + g_3) dx := I_0 + I_1 + I_2 + I_3,\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}I_0 :=& \int_{-\infty}^{\infty} \left[-\partial_x^j(p'(W)w_x)\partial_x^{j+1}(w + 2w_t) + (1 + \partial_t)\left(p'(W)(\partial_x^{j+1} w)^2\right)\right] dx \\ \leq& O(\delta + \delta_T)\left(\|w\|_{H^{j+1}}^2 + \|w_t\|_{H^j}^2\right).\end{aligned}$$

Here, we have used Lemmas 2.1, 2.2, 4.3. The rest terms on the right-hand side are estimated as follows.

$$I_1 \leq -\frac{d}{dt}\int_{-\infty}^{\infty} a_1(\partial_x^{j+1} w)^2\, dx$$

$$+ C(\delta_T+\delta)(\|w_t\|_{H^j}^2+\|w_x\|_{H^j}^2)+\alpha(\|\partial_x^{j+1}w\|^2+\|\partial_x^j w_t\|^2)+C\delta^2(1+t)^{-j-7/2}.$$

$$I_2 \le -\frac{d}{dt}\left(\int_{-\infty}^{\infty} a_2(\partial_x^{j+1}w)^2\,dx\right)+O(\delta_T+\delta)(\|w_t\|_{H^j}^2+\|w\|_{H^{j+1}}^2)$$
$$+\alpha(\|\partial_x^j w\|^2+\|\partial_x^j w_t\|^2)+O(\delta^2)(1+t)^{-j-7/2}.$$

$$I_3 \le O(\delta_T+\delta)(\|w_t\|_{H^j}^2+\|w\|_{H^{j+1}}^2).$$

The substitution of these estimates into (4.14) leads to

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|\partial_x^j w\|^2+\|\partial_x^j w_t\|^2+\frac{\varepsilon^2}{4}\|\partial_x^{j+2}w\|^2\right]\\
&\quad+\frac{d}{dt}\left[\int_{-\infty}^{\infty}\left(\partial_x^j w_t\cdot\partial_x^j w+(p'(W)+a_1+a_2+a_3)|\partial_x^{j+1}w|^2\right)\,dx\right]\\
&\quad+\frac{\varepsilon^2}{4}\|\partial_x^{j+2}w\|^2+(1-\alpha-O(\delta+\delta_T))\|\partial_x^j w_t\|^2\\
&\quad+\int_{-\infty}^{\infty}(p'(W)-\alpha-O(\delta+\delta_T))|\partial_x^{j+1}w|^2dx\\
&\le O(\delta_T+\delta)(\|w_t\|_{H^j}^2+\|w\|_{H^{j+1}}^2)+O(\delta^2)(1+t)^{-j-7/2},\quad j=1,2,3.
\end{aligned}\tag{4.15}$$

Integrating this inequality from 0 to t and taking summation of it (4.15) with respect to $j = 0, 1, 2, 3$, we get

$$\begin{aligned}
&\|w(t)\|_{H^5}^2+\|w_t(t)\|_{H^3}^2+\int_0^t(\|w_x(s)\|_{H^4}^2+\|w_t(s)\|_{H^3}^2)ds\\
\le& O(\delta_T+\delta)\int_0^t\left(\|w_t(s)\|_{H^3}^2+\|w_x(s)\|_{H^3}^2\right)\,ds\\
&+O(\delta+\delta_T)\int_0^t\|w(s)\|^2\,ds+O(\delta_0+\delta)^2+O(\delta_0+\delta)^2.
\end{aligned}\tag{4.16}$$

The first term on the right-hand side can be absorbed into the the damping energy and dissipation energy on the left-hand side. The term $\int_0^t\|w(s)\|^2\,ds = O(\delta+\delta_0)^2$ by (4.13). Thus, we obtain

$$\|w(t)\|_{H^5}^2+\|w_t(t)\|_{H^3}^2+\int_0^t(\|w_x(s)\|_{H^4}^2+\|w_t(s)\|_{H^3}^2)ds \le C(\delta_0+\delta)^2,\tag{4.17}$$

provided that $\delta_T+\delta$ is small enough.

Similarly, by performming

$$\int\sum_{j=0}^{3}\partial_t\partial_x^j(3.13)\cdot\partial_t\partial_x^j(w+2w_t)\,dx,$$

we can get

$$\|w_t(t)\|_{H^3}^2+\|w_{tt}(t)\|_{H^1}^2+\int_0^t\left(\|w_{tx}(s)\|_{H^2}^2+\|w_{tt}(s)\|_{H^1}^2\right)\,ds\le C(\delta_0+\delta)^2.\tag{4.18}$$

Combining (4.13), (4.17) and (4.18), we obtain the a priori energy estimate:

Theorem 4.11 *For local in-time solutions (z, w), it holds for $0 \le t \le T$ that*

$$\|z(t)\|^2 + \|z_t(t)\|^2 + \|w(t)\|^2_{H^5} + \|w_t(t)\|^2_{H^3} + \|w_{tt}\|^2_{H^1} + \int_0^t (\|z_t(s)\|^2 + \|w_x(s)\|^2_{H^4} + \|w_t(s)\|^2_{H^3} + \|w_{tt}(s)\|^2_{H^1}) ds \le C(\delta_0 + \delta)^2, \tag{4.19}$$

for $0 \le t \le T$, provided $\delta_T + \delta \ll 1$. Where

$$\delta_0 =: \|z(0)\| + \|z_t(0)\| + \|w(0)\|_{H^5} + \|w_t(0)\|_{H^3}.$$

4.4 Proof of global existence

Proof of Theorem 1.1: global existence. The local existence of (classical) solutions can be done by using the same argument as in [11]. The Theorem 4.11 shows that the local solutions satisfy the uniform bounds for (any) short time (and therefore satisfies (4.1) too) when initial perturbations are small enough. By using the continuous argument, we extend the local solution globally in time, which also satisfies Theorem 4.11 for any time. The proof is completed.

Proof of Theorem 1.2: global existence. From (1.24) and (1.25), we find $(S + \tau h(W))$ satisfies

$$(S + \tau h(W))_t + \frac{1}{2} u^2 + h(\rho) - h(W) = \frac{1}{2}\varepsilon^2 \frac{(\sqrt{\rho}\,)_{xx}}{\sqrt{\rho}} - \frac{1}{\tau}(S + \tau h(W)) + \tau h(W)_t,$$

where $W = W(\cdot + x_0, t)$. We express this equation in (w, η):

$$(S + \tau h(W))_t - \frac{1}{\tau}(S + \tau h(W)) = -\frac{(P(\tilde n)_x - \eta)^2}{(\tilde n + w)^4} - (h((\tilde n + w)^2) - h(\tilde n^2)) + \frac{1}{2}\varepsilon^2 \frac{(\tilde n + w)_{xx}}{\tilde n + w} + \tau h(W)_t.$$

Multiply above equation with $(S + \tau h(W))$ and integrate over $\mathbb{R}$. Using Theorem 1.1, Lemma 2.1 and Cauchy's inequality, we have

$$((S + \tau h(W))^2)_t - \frac{1}{2\tau}(S + \tau h(W))^2 \le O(1)(\|\eta\|^2 + \|w\|^2_{H^2}) + O(\delta^2)(1 + t)^{-3/2}$$

This leads to

$$\|S(\cdot, t) + \tau h(W(\cdot + x_0, t))\|^2 \le \|S_0 + \tau h\left(W\left(\cdot + x_0, 0\right)\right)\|^2 e^{-t/2\tau} + C\delta(1 + t)^{-3/2} + C(\|\eta\|^2 + \|w\|^2_{H^2}) \tag{4.20}$$

and

$$\|S(\cdot, t) + \tau h(W(\cdot + x_0, t))\|^2_{H^3} \le \|S_0 + \tau h\left(W\left(\cdot + x_0, 0\right)\right)\|^2_{H^3} e^{-t/2\tau} + C\delta^2(1 + t)^{-3/2} + C(\|\eta\|^2_{H^3} + \|w\|^2_{H^5}). \tag{4.21}$$

Thus, the proof is completed.

5 Time decay rate

5.1 A priori decay assumption and the main result

We shall use the idea in [22, 21] to obtain the explicit time decay rate for the global classical solutions and we need more estimates on higher order (both space and time) derivatives. It is not difficult to verify that Theorem 1.1–1.2 are also valid for solution with arbitrarily higher Sobolev regularity. In this section, we consider that the solutions satisfy

$$z \in C^k(0,\infty; H^{7-2k}),\ w \in C^k(0,\infty; H^{6-2k}), k = 0,1,2.$$

To perform a priori decay estimate, let us assume that for the global classical solution it holds a-priori that

$$\delta_T := \max_{0\le t\le T}\left[\|z(t)\| + (1+t)\|z_t(t)\| + \sum_{k=0}^{1}\sum_{j=0}^{5-2k}(1+t)^{(j+2k+1)/2}\|\partial_x^j\partial_t^k w(t)\| + (1+t)^3\|\partial_x^6 w(t)\|\right] \ll 1. \tag{5.1}$$

Notice that when $T = 0$,

$$\delta_0 = \|z(0)\| + \|z_t(0)\| + \sum_{k=0}^{1}\sum_{j=0}^{6-2k}\|\partial_t^k\partial_x^j w(0)\| \ll 1. \tag{5.2}$$

Under the assumption $\delta_0 \ll 1$, we can repeat the same argument in the previous section to get the existence of global classical solution with the following energy estimate

$$\begin{aligned}&\|z(t)\| + \|z_t(t)\| + \|w(t)\|_{H^6} + \|w_t(t)\|_{H^4} + \|w_{tt}\|_{H^2}\\ &\quad + \int_0^T(\|z_t(s)\| + \|w(s)\|_{H^6} + \|w_t(s)\|_{H^4} + \|w_{tt}(s)\|_{H^2})\,ds \le C(\delta_0+\delta).\end{aligned} \tag{5.3}$$

The main result in this section is

Theorem 5.1 *Under the assumption* (5.1)*, it holds for the global solutions* (z,w) *that*

$$\begin{aligned}&\sum_{k=0}^{2}\sum_{i=0}^{5-2k}\left[(1+t)^{i+2k+1}\|\partial_t^k\partial_x^i w(t)\|^2 + \int_0^t(1+s)^{i+2k}\|\partial_t^k\partial_x^i w(s)\|^2\,ds\right]\\ &+\sum_{k=0}^{3}\sum_{i=1}^{6-2k}\left[(1+t)^{i+2k}\|\partial_t^k\partial_x^i z(t)\|^2 + \int_0^t(1+s)^{i+2k-1}\|\partial_t^k\partial_x^i z(s)\|^2\,ds\right] \le O(\delta_0+\delta)^2,\end{aligned} \tag{5.4}$$

for $0 \le t \le T$*, provided* $\delta + \hat{\delta}_T$ *is small enough. Here* δ_0 *denotes the initial perturbation (5.2).*

Proof of Theorems 1.1–1.2: decay rate. In terms of the Sobolev Embedding theorem

$$\|f\|_{L^\infty} \le \|f\|^{1/2}\cdot\|f_x\|^{1/2}, \tag{5.5}$$

and (4.20)–(4.21), we can infer from Theorem 5.1 that

$$\|n(.,t) - \sqrt{W(.+x_0,t)}\,\|_{L^\infty} \le C(\delta_0+\delta)(1+t)^{-3/4}, \tag{5.6}$$

$$\|J(.,t) + \tau p\,(W(.+x_0,t))_x\,\|_{L^\infty} \le C(\delta_0+\delta)(1+t)^{-5/4}. \tag{5.7}$$

$$\|\,(S(.,t) + \tau h\,(W(.+x_0,t)))_x\,\|_{L^\infty} \le C(\delta_0+\delta)(1+t)^{-3/4}, \tag{5.8}$$

by which we complete the proofs of Theorems 1.1–1.2.
Strategy to prove Theorem 5.1: We shall obtain decay estimates through the following procedures:

$$P_z(k,j;i) =: \int_0^t(1+s)^i\int_{-\infty}^{\infty}\left[(\partial_t^k\partial_x^j(\text{z-equation}))\cdot(\partial_t^k\partial_x^j z)\right]\,dx\,ds \tag{5.9}$$

$$P_w(k,j;i) =: \int_0^t (1+s)^i \int_{-\infty}^{\infty} \left[(\partial_t^k \partial_x^j (\text{w-equation})) \cdot (\partial_t^k \partial_x^j w)\right] \, dx \, ds. \tag{5.10}$$

Let us define

$$\begin{aligned} N^2 =& \sum_{k=0}^{2} \sum_{i=0}^{5-2k} \left[(1+t)^{i+2k+1} \|\partial_t^k \partial_x^i w(t)\|^2 + \int_0^t (1+s)^{i+2k} \|\partial_t^k \partial_x^i w(s)\|^2 \, ds \right] \\ &+ \|z(t)\|^2 + \sum_{k=1}^{3} \left[(1+t)^{2k} \|\partial_t^k z(t)\|^2 + \int_0^t (1+s)^{2k-1} \|\partial_t^k z(s)\|^2 \, ds \right] \end{aligned} \tag{5.11}$$

¿From Lemma 5.3 below, N is equivalent to the right-hand side of (5.4). So, our goal is to show

$$N \le O(\delta + \delta_0). \tag{5.12}$$

5.2 Basic estimates

Lemma 5.2 *Under the assumption* (5.1), *it holds that for* $0 \le j + 2k \le 5$, $2 \le p \le \infty$

$$\|\partial_t^k \partial_x^j w(t)\|_{L^p} = O(\delta_T)(1+t)^{-3/4+1/2p-j/2-k} \text{ for } 0 \le t \le T. \tag{5.13}$$

Proof: For $k = 0, 1$, this basically follows from assumption 5.1. The estimate for w_{tt} follows from w-equation (3.13). □

¿From the relation $z_x = (2\tilde{n} + w)w$, and using (2.3) for $\tilde{n}$, assumption (5.1) and Lemma 5.2 for w, we can obtain the following relations between z_x and w.

Lemma 5.3 *Under the assumption* (5.1), *it holds that*

$$\|\partial_t^k \partial_x^j z_x\| = \sum_{l=0}^{k} \sum_{i=0}^{j} O(\delta_T + \delta)(1+t)^{(l-k)+(i-j)/2} \|\partial_t^l \partial_x^i w\|, \quad 0 \le 2k + j \le 5. \tag{5.14}$$

The assumption 5.1 also implies the following estimates for z, η and J.

Lemma 5.4 *Under the assumption* (5.1), *we have for* $0 \le t \le T$, $2 \le p \le \infty$,

$$\begin{aligned} &\|\partial_t^k \partial_x^j z(t)\|_{L^p} \le O(\delta_T + \delta)(1+t)^{-1/4+1/2p-j/2-k}, 0 \le j + 2k \le 6, \\ &\|\partial_t^k \partial_x^j \eta(t)\|_{L^p} \le O(\delta_T + \delta)(1+t)^{-5/4+1/2p-j/2-k}, 0 \le j + 2k \le 4, \\ &\|\partial_t^k \partial_x^j J(t)\|_{L^p} \le O(\delta_T + \delta)(1+t)^{-1/2+1/2p-j/2-k}, 0 \le j + 2k \le 4. \end{aligned} \tag{5.15}$$

Proof: The first estimate follows from Lemma 5.2 and assumption (5.1). The second estimate comes from $\eta = -z_t$. From $J = \eta - P(\tilde{n})_x$ and (2.3), we obtain the last estimate. □

Next, we use Lemmas 5.2 and 5.4 to give a priori estimates for the nonlinear terms f_i and g_i as the follows.

$$|f_1| = \left| \frac{1}{4}\varepsilon^2 \frac{(W_x + z_{xx})^2}{W + z_x} - p(W)_t - \frac{1}{4}\varepsilon^2 W_{xx} \right| \le O(\delta_T + \delta)(1+t)^{-1/2}|z_{xx}| + O(\delta) r_2.$$

$$|\partial_x f_1| \le O(\delta_T + \delta) \left[(1+t)^{-1/2}|z_{xxx}| + (1+t)^{-1}|z_{xx}| \right] + O(\delta) r_3$$

$$|\partial_t f_1| \le O(\delta_T + \delta) \left[(1+t)^{-1/2}|z_{xxt}| + (1+t)^{-1}|z_{xt}| + (1+t)^{-3/2}|z_{xx}| \right] + O(\delta) r_4$$

$$|f_2| = \left|\frac{(p(W)_x + z_t)^2}{W + z_x}\right| \le O(\delta + \delta_T)(1+t)^{-1/2}|z_t| + O(\delta) r_2$$

$$|\partial_x f_2| \le O(\delta + \delta_T)\left[(1+t)^{-1/2}|z_{tx}| + (1+t)^{-1}|z_{xx}| + (1+t)^{-1}|z_t|\right] + O(\delta) r_3$$

$$|\partial_t f_2| \le O(\delta + \delta_T)\left[(1+t)^{-1/2}|z_{tt}| + (1+t)^{-1}|z_{xt}| + (1+t)^{-3/2}|z_t|\right] + O(\delta) r_4$$

$$|f_3| = |p(W + z_x) - p(W) - p'(W) z_x| = O(|z_x|^2) \le O(\delta_T)(1+t)^{-1/2}|z_x|$$

$$|\partial_x f_3| \le O(\delta_T)(1+t)^{-1/2}|z_{xx}|$$
$$|\partial_t f_3| \le O(\delta_T)(1+t)^{-1/2}|z_{xt}|.$$

$$\begin{aligned}
g_1 &= \frac{(\tilde n_t + w_t)^2}{\tilde n + w} + \frac{\varepsilon^2}{4}\frac{(\tilde n_{xx} + w_{xx})^2}{\tilde n + w} - \frac{\varepsilon^2}{4}\tilde n_{xxxx} - \tilde n_{tt} \\
&= a_1 w_{xx} + O(1)(\delta + \hat\delta_T)(1+t)^{-1} w_t + O(1)\delta r_4 \qquad (5.16)\\
\partial_x g_1 &= a_1 w_{xxx} + O(\delta) r_5 \\
&\quad + O(\delta + \delta_T)\left[(1+t)^{-3/2} w_{xx} + (1+t)^{-2} w_x + (1+t)^{-3/2} w_t + (1+t)^{-1} w_{tx}\right] \\
\partial_t g_1 &= a_1 w_{xxt} + O(\delta) r_6 \\
&\quad + O(\delta + \delta_T)\left[(1+t)^{-2} w_{xx} + (1+t)^{-5/2} w_x + (1+t)^{-2} w_t + (1+t)^{-3/2} w_{tx} + (1+t)^{-1} w_{tt}\right]
\end{aligned}$$

$$\begin{aligned}
g_2 &= \frac{1}{2(\tilde n + w)}\left[\frac{(P(\tilde n)_x - \eta)^2}{(\tilde n + w)^2}\right]_{xx} \\
&= a_2 w_{xx} + b_2 w_{xt} + O(\delta) r_4 + O(\delta + \hat\delta_T)\left[(1+t)^{-3/2} w_x + (1+t)^{-1} w_t + (1+t)^{-2} w\right]
\end{aligned}$$

$$\begin{aligned}
\partial_x g_2 &= a_2 w_{xxx} + b_2 w_{xxt} + O(\delta) r_5 \\
&\quad + O(\delta + \hat\delta_T)\left[(1+t)^{-1} w_{xt} + (1+t)^{-3/2} w_{xx} + (1+t)^{-2} w_x + (1+t)^{-3/2} w_t + (1+t)^{-5/2} w\right]
\end{aligned}$$

$$\begin{aligned}
\partial_t g_2 &= a_2 w_{xxt} + b_2 w_{xtt} + O(\delta) r_6 \\
&\quad + O(\delta + \hat\delta_T)\left[(1+t)^{-2} w_{xx} + (1+t)^{-3/2} w_{xt} + (1+t)^{-1} w_{tt} + (1+t)^{-5/2} w_x + (1+t)^{-2} w_t + (1+t)^{-3} w\right]
\end{aligned}$$

$$g_3(x,t) = \left[(p'((\tilde n + w)^2) - p'(\tilde n^2))(\tilde n_x + w_x)\right]_x = a_3 w_{xx} + O(\delta + \delta_T)\left[(1+t)^{-1/2} w_x + (1+t)^{-1} w\right]$$

$$\partial_x g_3 = a_3 w_{xxx} + O(\delta + \delta_T)\left[(1+t)^{-1/2} w_{xx} + (1+t)^{-1} w_x + (1+t)^{-3/2} w\right]$$

$$\partial_t g_3 = a_3 w_{xxt} + O(\delta + \delta_T)\left[(1+t)^{-1/2} w_{xt} + (1+t)^{-3/2} w_x + (1+t)^{-1} w_t + (1+t)^{-2} w\right]$$

$$\partial_x [p'(W) w_x]_x = [p'(W) w_{xx}]_x + O(\delta)\left[(1+t)^{-1} w_x + (1+t)^{-1/2} w_{xx}\right]$$

$$\partial_t [p'(W) w_x]_x = [p'(W) w_{tx}]_x + O(\delta)\left[(1+t)^{-3/2} w_x + (1+t)^{-1/2} w_{xt}\right]$$

Here, we recall that

$$a_1 = \frac{\varepsilon^2}{4}\frac{2\tilde n_{xx} + w_{xx}}{\tilde n + w},\ a_2 = -\frac{J^2}{\rho^2},\ b_2 = \frac{2J}{\rho}, a_3 = (p'((\tilde n + w)^2) - p'(\tilde n^2).$$

and we have used

$$
\begin{aligned}
&\|a_1\|_\infty = O(\delta+\delta_T)(1+t)^{-1}, && \|a_2\|_\infty = O(\delta_T)(1+t)^{-1} \\
&\|b_2\|_\infty = O(\delta_T)(1+t)^{-1/2}, && \|a_3\|_\infty = O(\delta_T)(1+t)^{-3/4} \\
&\|a_{1,x}\|_\infty = O(\delta+\delta_T)(1+t)^{-3/2}, && \|a_{2,x}\|_\infty = O(\delta+\delta_T)(1+t)^{-3/2} \\
&\|b_{2,x}\|_\infty = O(\delta_T)(1+t)^{-1}, && \|a_{3,x}\|_\infty = O(\delta+\delta_T)(1+t)^{-5/4} \\
&\|a_{1,t}\|_\infty = O(\delta+\delta_T)(1+t)^{-2}, && \|a_{2,t}\|_\infty = O(\delta+\delta_T)(1+t)^{-2} \\
&\|b_{2,t}\|_\infty = O(\delta_T)(1+t)^{-3/2}, && \|a_{3,t}\|_\infty = O(\delta+\delta_T)(1+t)^{-7/4}.
\end{aligned}
$$

which follow from assumption (5.1). We summarize the above estimates as the following lemma.

Lemma 5.5 *Under the assumption (5.1), we have for $0 \le t \le T$, $2k+j \le 4$,*

$$|\partial_t^k \partial_x^j f_1| \le O(\delta+\delta_T) \sum_{l=0}^{k} \sum_{i=0}^{j+2k-2l} (1+t)^{-1/2-k+l-(j-i)/2} |\partial_t^l \partial_x^i z_{xx}| + O(\delta) r_{2+2k+j}$$

$$|\partial_t^k \partial_x^j f_2| \le O(\delta+\delta_T) \sum_{l=0}^{k} \sum_{i=0}^{j+2k-2l} (1+t)^{-1/2-k+l-(j-i)/2} |\partial_t^l \partial_x^i z_t| + O(\delta) r_{2+2k+j}$$

$$|\partial_t^k \partial_x^j f_3| \le O(\delta+\delta_T) \sum_{l=0}^{k} \sum_{i=0}^{j+2k-2l} (1+t)^{-1/2-k+l-(j-i)/2} |\partial_t^l \partial_x^i z_x|$$

$$\partial_t^k \partial_x^j g_1 = [a_1 \partial_t^k \partial_x^j w_x]_x + O(\delta) r_{4+2k+j} + O(\delta+\delta_T) \sum_{l=0}^{k} \sum_{i=0}^{j+2k-2l} (1+t)^{-1-k+l-(j-i)/2} \partial_t^l \partial_x^i w_t$$

$$\partial_t^k \partial_x^j g_2 = [a_2 \partial_t^k \partial_x^j w_x]_x + [b_2 \partial_t^k \partial_x^j w_t]_x + O(\delta) r_{4+2k+j} + O(\delta+\delta_T) \sum_{l=0}^{k+1} \sum_{i=0}^{j+2k-2l+1} (1+t)^{-2-k+l-(j-i)/2} \partial_t^l \partial_x^i w$$

$$\partial_t^k \partial_x^j g_3 = [a_3 \partial_t^k \partial_x^j w_x]_x + O(\delta+\delta_T) \sum_{l=0}^{k} \sum_{i=0}^{j+2k-2l+1} (1+t)^{-1-k+l-(j-i)/2} \partial_t^l \partial_x^i w$$

$$\partial_t^k \partial_x^j [p'(W) w_x]_x = [p'(W) \partial_t^k \partial_x^j w_x]_x + O(\delta+\delta_T) \sum_{l=0}^{k} \sum_{i=0}^{j+2k-2l} (1+t)^{-1/2-k+l-(j-i)/2} \partial_t^l \partial_x^i w_x$$

5.3 Decay estimates for w

We have seen that by performming $\sum_{0\le j\le 4} P_w(0,j;0) + P_z(0,0;0)$, we have got the following energy estimate for (z,w):

Proposition 5.6 *Under the assumption (5.1), it holds for the global solution (z,w) that*

$$
\begin{aligned}
&\|z(t)\|_{H^2}^2 + \|z_t(t)\|^2 + \|w(t)\|_{H^6}^2 + \|w_t(t)\|_{H^4}^2 \\
&\int_0^t (\|z_x(s)\|_{H^1}^2 + \|z_t(s)\|^2 + \|w(s)\|_{H^6}^2 + \|w_t(s)\|_{H^4}^2)\, ds \le C(\delta_0+\delta)^2,
\end{aligned}
\tag{5.17}
$$

provided $\delta_0 + \delta \ll 1$.

The integral part of (5.17) will be used for the next-order decay estimate.

Proposition 5.7 *Under the assumption* (5.1), *it holds for the global solutions w that*

$$\sum_{j=0}^{4}(1+t)^{j+1}\left(\|\partial_x^j w(t)\|_{H^2}^2+\|\partial_x^j w_t(t)\|^2\right)$$
$$+\sum_{j=0}^{4}\int_0^t(1+s)^{j+1}\left(\|\partial_x^{j+1}w(s)\|_{H^1}^2+\|\partial_x^j w_t(s)\|^2\right)ds\le O(N_1),\tag{5.18}$$

provided that $\delta+\hat{\delta}_T$ is small enough. Where

$$N_1:=(\delta+\delta_0)^2+(\delta+\delta_T)\delta_T^2.\tag{5.19}$$

Proof: We perform $\int_{-\infty}^{\infty}\partial_x^j(3.13)\cdot\partial_x^j(w+2w_t)\,dx$ for $j=0,...,4$. Using integration by part, we obtain

$$\frac{d}{dt}\left(\|\partial_x^j w_t\|^2+\frac{1}{2}\|\partial_x^j w\|^2+\int_{-\infty}^{\infty}p'(W)(\partial_x^{j+1}w)^2\,dx+\frac{\varepsilon^2}{4}\|\partial_x^{j+2}w\|^2+\int_{-\infty}^{\infty}\partial_x^j w_t\cdot\partial_x^{j+1}w\,dx\right)$$
$$+\left(\frac{\varepsilon^2}{4}\|\partial_x^{j+2}w\|^2+\|\partial_x^j w_t\|^2+\int_{-\infty}^{\infty}p'(W)(\partial_x^{j+1}w)^2\,dx\right)$$
$$\le\int_{-\infty}^{\infty}\left(\partial_x^j(g_1+g_2+g_3)\cdot\partial_x^j(w+2w_t)\right)dx+I_0+I_1,\tag{5.20}$$

where

$$I_0:=\int_{-\infty}^{\infty}\left[\left(p'(W)\partial_x^{j+1}w-\partial_x^j(p'(W)w_x)\right)\cdot(\partial_x^{j+1}w)\right]dx,$$
$$I_1:=\int_{-\infty}^{\infty}\left[\partial_t\left(p'(W)(\partial_x^{j+1}w)^2\right)-2\partial_x^j(p'(W)w_x)\cdot\partial_x^{j+1}w_t\right]dx.\tag{5.21}$$

By using Lemma 5.5, the terms on the right-hand side of (5.20) are estimated as the follows.

$$|I_0|\le O(\delta)\|\partial_x^{j+1}w\|^2+O(\delta)\sum_{i=1}^{j}(1+t)^{-1-j+i}\|\partial_x^i w\|^2$$
$$|I_1|=\int_{-\infty}^{\infty}\left|p'(W)_t(\partial_x^{j+1}w)^2-2\left[p'(W)\partial_x^j w_x-\partial_x^j(p'(W)w_x)\right]_x\cdot\partial_x^j w_t\right|dx$$
$$\le O(\delta)(1+t)^{-1}\|\partial_x^{j+1}w\|^2+\alpha\|\partial_x^j w_t\|^2+\left\|[p'(W)\partial_x^j w_x-\partial_x^j(p'(W)w_x)]_x\right\|^2$$
$$\le O(\delta)(1+t)^{-1}\|\partial_x^{j+1}w\|^2+\alpha\|\partial_x^j w_t\|^2+O(\delta)\sum_{i=1}^{j+1}(1+t)^{-2-j+i}\|\partial_x^i w\|^2$$
$$\le\alpha\|\partial_x^j w_t\|^2+O(\delta)\sum_{i=1}^{j+1}(1+t)^{-2-j+i}\|\partial_x^i w\|^2\tag{5.22}$$

$$\int_{-\infty}^{\infty}\partial_x^j(g_1+g_2)\cdot\partial_x^j w\,dx\le-\int_{-\infty}^{\infty}(a_1+a_2)(\partial_x^{j+1}w)^2\,dx+\alpha(1+t)^{-1}\|\partial_x^j w\|^2$$
$$+(1+t)\left[O(\delta+\delta_T)^2\sum_{l=0}^{1}\sum_{i=0}^{j+1-2l}(1+t)^{-4+2l-j+i}\|\partial_t^l\partial_x^i w\|^2+O(\delta^2)\|r_{4+j}\|^2\right]$$
$$\le-\int_{-\infty}^{\infty}(a_1+a_2)(\partial_x^{j+1}w)^2\,dx+\alpha(1+t)^{-1}\|\partial_x^j w\|^2$$

$$+ O(\delta+\delta_T)^2 \sum_{l=0}^{1} \sum_{i=0}^{j+1-2l} (1+t)^{-3+2l-j+i} \|\partial_t^l \partial_x^i w\|^2 + O(\delta^2)(1+t)^{-5/2-j}$$

$$\int_{-\infty}^{\infty} \partial_x^j (g_1+g_2)\cdot 2\partial_x^j w_t \, dx \le -\frac{d}{dt}\left(\int_{-\infty}^{\infty} (a_1+a_2)(\partial_x^{j+1} w)^2 \, dx\right) + \alpha \|\partial_x^j w_t\|^2 + O(1)\|g_1+g_2\|^2$$

$$\le -\frac{d}{dt}\left(\int_{-\infty}^{\infty} (a_1+a_2)(\partial_x^{j+1} w)^2 \, dx\right) + \alpha \|\partial_x^j w_t\|^2$$

$$+ O(\delta+\delta_T)^2 \sum_{l=0}^{1} \sum_{i=0}^{j+1-2l} (1+t)^{-4+2l-j+i} \|\partial_t^l \partial_x^i w\|^2 + O(\delta^2)(1+t)^{-7/2-j} \tag{5.23}$$

$$\int_{-\infty}^{\infty} \partial_x^j g_3 \cdot \partial_x^j w \, dx \le -\int_{-\infty}^{\infty} a_3 (\partial_x^{j+1} w)^2 \, dx$$

$$+ O(\delta+\delta_T)(1+t)^{-1}\|\partial_x^j w\|^2 + O(\delta+\delta_T)(1+t) \sum_{1\le i\le j+1} (1+t)^{-2-j+i} \|\partial_x^i w\|^2$$

$$\le -\int_{-\infty}^{\infty} a_3 (\partial_x^{j+1} w)^2 \, dx + O(\delta+\delta_T) \sum_{i=0}^{j+1} (1+t)^{-1-j+i} \|\partial_x^i w\|^2\|^2$$

$$\int_{-\infty}^{\infty} \partial_x^j g_3 \cdot 2\partial_x^j w_t \, dx \le -\frac{d}{dt}\left(\int_{-\infty}^{\infty} a_3 (\partial_x^{j+1} w)^2 \, dx\right) + \alpha \|\partial_x^j w_t\|^2 + O(\delta+\delta_T)^2 \sum_{i=0}^{j+1} (1+t)^{-2-j+i} \|\partial_x^i w\|^2. \tag{5.24}$$

Hence, we obtain for $j = 0, ..., 4$

$$\frac{d}{dt}\left(\|\partial_x^j w_t\|^2 + \frac{1}{2}\|\partial_x^j w\|^2 + \int_{-\infty}^{\infty} (p'(W) - O(\delta+\delta_T))\,(\partial_x^{j+1} w)^2 \, dx + \frac{\varepsilon^2}{4}\|\partial_x^{j+2} w\|^2 + \int_{-\infty}^{\infty} \partial_x^j w_t \cdot \partial_x^{j+1} w \, dx\right)$$

$$+ \left(\frac{\varepsilon^2}{4}\|\partial_x^{j+2} w\|^2 + (1-\alpha - O(\delta+\delta_T))\|\partial_x^j w_t\|^2 + \int_{-\infty}^{\infty} (p'(W) - \alpha - O(\delta_T+\delta))\,(\partial_x^{j+1} w)^2 \, dx\right)$$

$$\le O(\delta+\delta_T+\alpha) \sum_{i=0}^{j} (1+t)^{-1-j+i} \|\partial_x^i w\|^2 + O(\delta+\delta_T) \sum_{i=0}^{j-1} (1+t)^{-1-j+i} \|\partial_x^i w_t\|^2 + O(\delta^2)(1+t)^{-5/2-j}. \tag{5.25}$$

Now, take $j = 0$ in the above equation. We perform $\int_0^t (1+s)^i (5.25)_{j=0}\, ds$ for $i = 1$. This yields

$$(1+t)(\|w(t)\|_{H^2}^2 + \|w_t(t)\|^2) + \int_0^t (1+s)(\|w_x(s)\|_{H^1}^2 + \|w_t(s)\|^2)\, ds$$

$$\le O(\delta_0+\delta)^2 + \int_0^t (\|w_x(s)\|_{H^1}^2 + \|w_t(s)\|^2)\, ds + O(\delta+\delta_T+\alpha)\int_0^t \|w(s)\|^2\, ds \le O(N_1). \tag{5.26}$$

Here, the last step follows from (5.17).

Next, we perform $\int_0^t (1+s)(5.25)_{j=1}\, ds$. When $i = 1$, using (5.17), we get

$$(1+t)\left[\|w_x(t)\|_{H^2}^2 + \|w_{xt}(t)\|^2\right] + \int_0^t (1+s)(\|w_{xx}(s)\|_{H^1}^2 + \|w_{xt}(s)\|^2)\, ds$$

$$\le O(\delta+\delta_0)^2 + \int_0^t (\|w_x(s)\|_{H^2}^2 + \|w_{xt}(s)\|^2)\, ds + \int_0^t \left[O(\delta+\delta_T+\alpha)\left(\|w_x(s)\|^2 + (1+s)^{-1}\|w(s)\|^2\right)\right] ds$$

$$\le O(N_1).$$

Now, we combine this with the result in (5.26) to get

$$\int_0^t (1+s)(\|w_x(s)\|_{H^2}^2 + \|w_{xt}(s)\|^2)\,ds \le O(N_1). \tag{5.27}$$

Thirdly, we perform $\int_0^t (1+s)^2 (5.25)_{j=1}\,ds$. Using (5.27), we obtain

$$(1+t)^2 \left(\|w_x(t)\|_{H^2}^2 + \|w_{xt}\|^2\right) + \int_0^t (1+s)^2(\|w_{xx}(s)\|_{H^1}^2 + \|w_{xt}(s)\|^2)\,ds \le O(N_1).$$

For $j = 2, 3, 4$, through the same procedures

$$\int_0^t (1+s)^i (5.25)_j\,ds$$

for $i = 1, ..., j+1$, we can inductively obtain that for $j = 2, 3, 4$

$$(1+t)^{j+1}(\|\partial_x^j w(t)\|_{H^2}^2 + \|\partial_x^j w_t(t)\|^2) + \int_0^t (1+s)^{j+1}(\|\partial_x^j w_x(s)\|_{H^1}^2 + \|\partial_x^j w_t(s)\|^2)\,ds \le O(N_1).$$

□

Proposition 5.8 *Under the assumption* (5.1), *it holds for the global solutions w that for $j = 0, ..., 4$, $0 \le t \le T$,*

$$(1+t)^{j+2}(\|\partial_x^j w_t(t)\|^2 + \|\partial_x^{j+1} w(t)\|_{H^1}^2) + \int_0^t (1+s)^{j+2}\|\partial_x^j w_t(s)\|^2\,ds \le O(N_1), \tag{5.28}$$

provided $\delta + \hat{\delta}_T$ is small enough. Where N_1 is defined by (5.19).

Proof: For $j = 0, ..., 4$, we perform

$$\int_{-\infty}^{\infty} \partial_x^j (3.13) \cdot 2\partial_x^j w_t\,dx.$$

By integration by parts, we obtain

$$\frac{d}{dt}\left(\|\partial_x^j w_t\|^2 + \int_{-\infty}^{\infty} p'(W)(\partial_x^{j+1} w)^2\,dx + \frac{\varepsilon^2}{4}\|\partial_x^{j+2} w\|^2\right) + \|\partial_x^j w_t\|^2 \;\le \alpha\|\partial_x^j w_t\|^2 + \sum_{m=0}^{3}\|\partial_x^j g_m\|^2 + I_1$$

where I_1 is defined by (5.21). We use (5.22), (5.23), (5.24) and

$$\int_{-\infty}^{\infty} \delta r_{4+j}\partial_x^j w_t\,dx \le \alpha\|\partial_x^j w_t\|^2 + O(\delta^2)(1+t)^{-7/2-j}$$

to get

$$\begin{aligned}&\frac{d}{dt}\left[\|\partial_x^j w_t\|^2 + \int_{-\infty}^{\infty}(p'(W) + a_1 + a_2 + a_3)(\partial_x^{j+1} w)^2 + \frac{\varepsilon^2}{4}\|\partial_x^{j+2} w\|^2\right] + (1 - \alpha - O(\delta + \delta_T))\|\partial_x^j w_t\|^2\\ \le\, & O(\delta + \delta_T)\left(\sum_{i=0}^{j+1}(1+t)^{-2-j+i}\|\partial_x^i w\|^2 + \sum_{i=0}^{j-1}(1+t)^{-2-j+i}\|\partial_x^i w_t\|^2\right) + O(\delta^2)(1+t)^{-7/2-j}.\end{aligned} \tag{5.29}$$

Multiplying this equation by $(1+s)^{j+2}$ and integrating it from 0 to t, using Proposition 5.7, we obtain (5.28).

□

Similar to the procedures to estimate w and its x-derivatives, we perform the procedures

$$\int_{-\infty}^{\infty} \partial_x^j \partial_t^k (\text{w-equation}) \cdot \partial_x^j \partial_t^k (w + 2w_t)\, dx \text{ for } 0 \le j + 2k \le 4$$

and

$$\int_{-\infty}^{\infty} \partial_x^j \partial_t^k (\text{w-equation}) \cdot \partial_x^j \partial_t^k 2w_t\, dx \text{ for } 0 \le j + 2k \le 4.$$

These lead to the following proposition. Its proof is omitted.

Proposition 5.9 *Under the assumption* (5.1)*, it holds for the global solutions w, $0 \le j+2k \le 4$, that*

$$\begin{aligned}&(1+t)^{2k+j+1}\left(\|\partial_t^k \partial_x^j w(t)\|_{H^2}^2 + \|\partial_t^k \partial_x^j w_t(t)\|^2\right)\\&+ \int_0^t (1+s)^{2k+j+1}\left(\|\partial_t^k \partial_x^{j+1} w(s)\|_{H^1}^2 + \|\partial_t^k \partial_x^j w_t(s)\|^2\right) ds \le O(N_1),\end{aligned} \tag{5.30}$$

$$(1+t)^{2k+j+2}\left(\|\partial_t^k \partial_x^j w_t(t)\|^2 + \|\partial_t^k \partial_x^j w_x(t)\|_{H^1}^2\right) + \int_0^t (1+s)^{2k+j+2}\|\partial_t^k \partial_x^j w_t(s)\|^2\, ds \le O(N_1), \tag{5.31}$$

provided that $\delta + \hat{\delta}_T$ is small enough. Here,

$$N_1 := (\delta + \delta_0)^2 + (\delta + \delta_T)\delta_T^2.$$

5.4 Decay estimates for z_t, z_{tt} and z_{ttt}

Proposition 5.10 *Under the assumption* (5.1)*, it holds for the global solutions z that for $k = 0, 1, 2$,*

$$(1+t)^{2k}\left(\|\partial_t^k z(t)\|_{H^2}^2 + \|\partial_t^k z_t(t)\|^2\right) + \int_0^t (1+s)^{2k}\left(\|\partial_t^k z_x(s)\|_{H^1}^2 + \|\partial_t^k z_t(s)\|^2\right) ds \le O(N_1), \tag{5.32}$$

$$(1+t)^{2k+1}\left(\|\partial_t^k z_x(t)\|_{H^1}^2 + \|\partial_t^k z_t(t)\|^2\right) + \int_0^t (1+s)^{2k+1}\|\partial_t^k z_t(s)\|^2\, ds \le O(N_1), \tag{5.33}$$

for $0 \le t \le T$, provided that $\delta + \hat{\delta}_T$ is small enough.

Proof: By performming the procedure

$$\int_{-\infty}^{\infty} \partial_t^k (\text{z-equation}) \cdot \partial_t^k (z + 2z_t)\, dx \text{ and } \int_{-\infty}^{\infty} \partial_t^k (\text{z-equation}) \cdot \partial_t^k 2z_t\, dx$$

for $k = 0, 1, 2$ and integrating by part, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t^k z_t\|^2 + \frac{1}{2}\|\partial_t^k z\|^2 + \int_{-\infty}^{\infty} p'(W)(\partial_t^k z_x)^2\, dx + \frac{\varepsilon^2}{4}\|\partial_t^k z_{xx}\|^2 + \int_{-\infty}^{\infty} \partial_t^k z_x \cdot \partial_t^k z_t\, dx\right)\\&+ \|\partial_t^k z_t\|^2 + \frac{\varepsilon^2}{4}\|\partial_t^k z_{xx}\|^2 + \int_{-\infty}^{\infty} p'(W)(\partial_t^k z_x)^2\, dx\\&= \int_{-\infty}^{\infty}\left(\sum_{m=0}^{3} \partial_t^k f_{m,x} \cdot \partial_t^k (z + 2z_t)\right) dx + J_0 + J_1,\end{aligned} \tag{5.34}$$

and

$$\frac{d}{dt}\left(\|\partial_t^k z_t\|^2 + \int_{-\infty}^{\infty} p'(W)(\partial_t^k z_x)^2\, dx + \frac{\varepsilon^2}{4}\|\partial_t^k z_{xx}\|^2\right) + \|\partial_t^k z_t\|^2$$

$$= \int_{-\infty}^{\infty} \left(\sum_{m=0}^{3} \partial_t^k f_{m,x} \cdot \partial_t^k 2z_t \right) dx + J_1, \tag{5.35}$$

where

$$J_0 := \int_{-\infty}^{\infty} \left[\left(p'(W)\partial_t^k z_x - \partial_t^k (p'(W)z_x) \right) \cdot (\partial_t^k z_x) \right] dx,$$
$$J_1 := \int_{-\infty}^{\infty} \left[\partial_t \left(p'(W)(\partial_t^k z_x)^2 \right) - 2\partial_t^k (p'(W)z_x) \cdot (\partial_t^k z_{xt}) \right] dx. \tag{5.36}$$

By using Lemma 5.5, we get

$$|J_0| \le O(\delta)\|\partial_t^k z_x\|^2 + O(\delta) \sum_{l=0}^{k-1} (1+t)^{-2k+2l} \|\partial_t^l z_x\|^2 \le O(\delta) \sum_{l=0}^{k} (1+t)^{-2k+2l} \|\partial_t^l w\|^2$$

$$|J_1| = \int_{-\infty}^{\infty} \left| p'(W)_t (\partial_t^k z_x)^2 - 2\left[p'(W)\partial_t^k z_x - \partial_t^k (p'(W)z_x) \right] \cdot \partial_t^k z_{tx} \right| dx$$
$$\le O(\delta)(1+t)^{-1}\|\partial_t^k z_x\|^2 + O(\delta)(1+t)\|\partial_t^{k+1} z_x\|^2 + \delta^{-1}(1+t)^{-1} \left\| p'(W)\partial_t^k z_x - \partial_t^k (p'(W)z_x) \right\|^2$$
$$\le O(\delta)(1+t)^{-1}\|\partial_t^k z_x\|^2 + O(\delta)(1+t)\|\partial_t^{k+1} z_x\|^2 + O(\delta) \sum_{l=0}^{k-1} (1+t)^{-1-2k+2l} \|\partial_t^l z_x\|^2$$
$$\le O(\delta) \sum_{l=0}^{k+1} (1+t)^{-1-2k+2l} \|\partial_t^l w\|^2$$

$$\int_{-\infty}^{\infty} \partial_t^k (f_1 + f_2 + f_3) \cdot \partial_t^k z_x \, dx \le \|\partial_t^k z_x\|^2 + O(1) \left(\|\partial_t^k f_1\|^2 + \|\partial_t^k f_2\|^2 + \|\partial_t^k f_3\|^2 \right)$$

$$\int_{-\infty}^{\infty} \partial_t^k (f_1 + f_2 + f_3) \cdot \partial_t^k z_{tx} \, dx \le (1+t)\|\partial_t^{k+1} z_x\|^2 + O(1)(1+t)^{-1} \left(\|\partial_t^k f_1\|^2 + \|\partial_t^k f_2\|^2 + \|\partial_t^k f_3\|^2 \right)$$
$$\le (\delta + \delta_T) \sum_{l=0}^{k+1} (1+t)^{-1-2k+2l} \|\partial_t^l w\|^2 + O(1)(1+t)^{-1} \left(\|\partial_t^k f_1\|^2 + \|\partial_t^k f_2\|^2 + \|\partial_t^k f_3\|^2 \right)$$

$$\|\partial_t^k f_1\|^2 \le O(\delta + \delta_T) \sum_{l=0}^{k} \sum_{i=0}^{2k-2l} (1+t)^{-1-2k+2l+i} \|\partial_t^l \partial_x^i z_{xx}\|^2 + O(\delta^2)\|r_{2+2k}\|^2$$
$$\le O(\delta + \delta_T) \sum_{l=0}^{k} \sum_{i=1}^{1+2k-2l} (1+t)^{-2-2k+2l+i} \|\partial_t^l \partial_x^i w\|^2 + O(\delta^2)(1+t)^{-3/2-2k}$$

$$\|\partial_t^k f_2\|^2 \le O(\delta + \delta_T) \sum_{l=0}^{k} \sum_{i=0}^{2k-2l} (1+t)^{-1-2k+2l+i} \|\partial_t^l \partial_x^i z_t\|^2 + O(\delta^2)\|r_{2+2k}\|^2$$
$$\le O(\delta + \delta_T) \sum_{l=0}^{k} (1+t)^{-1-2k+2l} \|\partial_t^l z_t\|^2 + \sum_{l=1}^{k+1} \sum_{i=0}^{2k-2l-1} (1+t)^{-2-2k+2l+i} \|\partial_t^l \partial_x^i z_x\|^2 + O(\delta^2)(1+t)^{-3/2-2k}$$
$$\le O(\delta + \delta_T) \sum_{l=1}^{k+1} (1+t)^{-3-2k+2l} \|\partial_t^l z\|^2 + \sum_{l=1}^{k+1} \sum_{i=0}^{2k-2l-1} (1+t)^{-2-2k+2l+i} \|\partial_t^l \partial_x^i w\|^2 + O(\delta^2)(1+t)^{-3/2-2k}$$

$$\|\partial_t^k f_3\|^2 \le O(\delta+\delta_T)\sum_{l=0}^{k}\sum_{i=0}^{2k-2l}(1+t)^{-1-2k+2l+i}\|\partial_t^l\partial_x^i z_x\|^2 \le O(\delta+\delta_T)\sum_{l=0}^{k}\sum_{i=0}^{2k-2l}(1+t)^{-1-2k+2l+i}\|\partial_t^l\partial_x^i w\|^2.$$

Putting all these together, (5.34) becomes

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t^k z_t\|^2+\frac{1}{2}\|\partial_t^k z\|^2+\int_{-\infty}^{\infty}p'(W)(\partial_t^k z_x)^2\,dx+\frac{\varepsilon^2}{4}\|\partial_t^k z_{xx}\|^2+\int_{-\infty}^{\infty}\partial_t^k w_x\cdot\partial_t^k w_t\,dx\right)\\
&+(1-O(\delta+\delta_T))\|\partial_t^{k+1}z\|^2+\frac{\varepsilon^2}{4}\|\partial_t^k z_{xx}\|^2+\int_{-\infty}^{\infty}\left(p'(W)-O(\delta)\right)(\partial_t^k z_x)^2\,dx\\
&\le O(\delta+\delta_T)\left[\sum_{l=0}^{k}(1+t)^{-2k+2l}\|\partial_t^l w\|^2+(1+t)\|\partial_t^{k+1}w\|^2+\sum_{l=0}^{k}\sum_{i=1}^{2k-2l}(1+t)^{-1-2k+2l+i}\|\partial_t^l\partial_x^i w\|^2\right.\\
&\quad\left.+\sum_{l=0}^{k}(1+t)^{-1}\|\partial_t^l\partial_x^{2k-2l+1}w\|^2\right]+O(\delta+\delta_T)\sum_{l=1}^{k}(1+t)^{-3-2k+2l}\|\partial_t^l z\|^2+O(\delta^2)(1+t)^{-3/2-2k}.
\end{aligned}\tag{5.37}$$

And (5.35) becomes

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t^k z_t\|^2+\int_{-\infty}^{\infty}p'(W)(\partial_t^k z_x)^2\,dx+\frac{\varepsilon^2}{4}\|\partial_t^k z_{xx}\|^2\right)+\|\partial_t^k z_t\|^2\\
&\le O(\delta+\delta_T)\left[\sum_{l=0}^{k+1}(1+t)^{-2k-1+2l}\|\partial_t^l w\|^2+\sum_{l=0}^{k}\sum_{i=1}^{2k-2l}(1+t)^{-2-2k+2l+i}\|\partial_t^l\partial_x^i w\|^2\right.\\
&\quad\left.+\sum_{l=0}^{k}(1+t)^{-2}\|\partial_t^l\partial_x^{2k-2l+1}w\|^2\right]\\
&+O(\delta+\delta_T)\sum_{l=1}^{k}(1+t)^{-4-2k+2l}\|\partial_t^l z\|^2+O(\delta^2)(1+t)^{-5/2-2k}.
\end{aligned}\tag{5.38}$$

We notice that from Theorem

$$\int_0^t(1+s)^{2k}\text{right-hand-side of (5.37)}\,ds=O(N_1)$$

$$\int_0^t(1+s)^{2k+1}\text{right-hand-side of (5.38)}\,ds=O(N_1)$$

Using these, we proceed the following procedures:

- For $k=0$, $\int_0^t(5.37)_{k=0}\,ds$ leads to (5.39) below;
- For $k=0$, $\int_0^t(1+s)(5.38)_{k=0}\,ds$ leads to (5.40);
- For $k=1$, $\int_0^t\sum_{i=0}^{2k}(1+s)^i(5.37)_{k=1}\,ds$ leads to (5.41);
- For $k=1$, $\int_0^t(1+s)^{2k+1}(5.37)_{k=1}\,ds$ leads to (5.42);
- For $k=2$, $\int_0^t\sum_{i=0}^{2k}(1+s)^i(5.37)_{k=2}\,ds$ leads to (5.43);
- For $k=2$, $\int_0^t(1+s)^{2k+1}(5.37)_{k=2}\,ds$ leads to (5.44).

$$\|z(t)\|_{H^2}^2 + \|z_t(t)\|^2 + \int_0^t (\|z_x(s)\|_{H^1}^2 + \|z_t(s)\|^2)\, ds = O(N_1) \tag{5.39}$$

$$(1+t)(\|z_x(t)\|_{H^1}^2 + \|z_t(t)\|^2) + \int_0^t (1+s)\|z_t(s)\|^2\, ds = O(N_1) \tag{5.40}$$

$$(1+t)^2 \left(\|z_t(t)\|_{H^2}^2 + \|z_{tt}(t)\|^2\right) + \int_0^t (1+s)^2(\|z_{tx}(s)\|_{H^1}^2 + \|z_{tt}(s)\|^2)\, ds = O(N_1) \tag{5.41}$$

$$(1+t)^3(\|z_{tx}(t)\|_{H^1}^2 + \|z_{tt}(t)\|^2) + \int_0^t (1+s)^3\|z_{tt}(s)\|^2\, ds = O(N_1) \tag{5.42}$$

$$(1+t)^4 \left(\|z_{tt}(t)\|_{H^2}^2 + \|z_{ttt}(t)\|^2\right) + \int_0^t (1+s)^4(\|z_{ttx}(s)\|_{H^1}^2 + \|z_{ttt}(s)\|^2)\, ds = O(N_1) \tag{5.43}$$

$$(1+t)^5(\|z_{ttx}(t)\|_{H^1}^2 + \|z_{ttt}(t)\|^2) + \int_0^t (1+s)^5\|z_{ttt}(s)\|^2\, ds = O(N_1). \tag{5.44}$$

This completes the proof. □

In general, we have the following proposition. Its proof is the same as Proposition 5.10. We shall not repeat it.

Proposition 5.11 *Under the assumption* (5.1)*, it holds for the global solutions z that for $0 \le j+2k \le 4$,*

$$(1+t)^{2k+j}\left(\|\partial_t^k\partial_x^j z(t)\|_{H^2}^2 + \|\partial_t^k\partial_x^j z_t(t)\|^2\right) + \int_0^t (1+s)^{2k+j}\left(\|\partial_t^k\partial_x^j z_x(s)\|_{H^1}^2 + \|\partial_t^k\partial_x^j z_t(s)\|^2\right) ds \le O(N_1),$$

$$(1+t)^{2k+j+1}\left(\|\partial_t^k\partial_x^j z_x(t)\|_{H^1}^2 + \|\partial_t^k\partial_x^j z_t(t)\|^2\right) + \int_0^t (1+s)^{2k+j+1}\|\partial_t^k\partial_x^j z_t(s)\|^2\, ds \le O(N_1),$$

for $0 \le t \le T$, provided that $\delta + \hat{\delta}_T$ is small enough.

Proof of Theorem 5.1: Recall

$$\begin{aligned} N^2 :=& \sum_{k=0}^{2}\sum_{i=0}^{5-2k}\left[(1+t)^{i+2k+1}\|\partial_t^k\partial_x^i w(t)\|^2 + \int_0^t (1+s)^{i+2k}\|\partial_t^k\partial_x^i w(s)\|^2\, ds\right] \\ &+ \|z(t)\|^2 + \sum_{k=1}^{2}\left[(1+t)^{2k}\|\partial_t^k z(t)\|^2 + \int_0^t (1+s)^{2k-1}\|\partial_t^k z(s)\|^2\, ds\right] \end{aligned} \tag{5.45}$$

and $N_1 = O(\delta+\delta_0)^2 + (\delta+\delta_T)\delta_T^2$. Combining all estimates in this section, we get

$$N^2 = O(N_1) \le O(\delta+\delta_0)^2 + (\delta+N)N^2.$$

When $\delta + \delta_0$ is small enough, we can get $N \le O(\delta+\delta_0)$.

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References

[1] L. Brüll and H. Lange, The Schrödinger-Langevin equation: Special solutions and nonexistence of solitary waves, *J. Math. Phys.* **25** (1984), 786–790.

[2] I-L. Chern, Multiple-mode diffusion waves for viscous non-strictly hyperbolic conservation laws, *Comm. Math. Phys.* **138** (1991), no. 1, 51–61.

[3] I-L. Chern and T.-P. Liu, Convergence to diffusion waves of solutions for viscous conservation laws, *Comm. Math. Phys.* **110** (1987), no. 3, 503–517.

[4] H. Dekker, Classical and quantum mechanics of damped harmonic oscillator, *Physics Reports,* **80(1)** (1981), 1-112.

[5] H.-D. Doebner, G.A. Goldin, and P. Nattermann, Gauge transformations in quantum mechanics and the unification of nonlinear Schrödinger equations *J. Math. Phys.* **40** (1999), 49-63.

[6] C. T. Duyn and L. A. Peletier, A class of similarity solutions of the nonlinear diffusion equation, *Nonlinear Analysis, TMA* **1** (1977), 223–233.

[7] G. W. Ford, M. Kac and P Mazur, Statistical mechanics of assemblies of coupled oscillators, *J. Math. Phys.* **6** (1965), 504–515.

[8] G. W. Ford and M. Kac, On the quantum Langevin equation, *J. Statist. Phys.* **46** (1987), no. 5-6, 803–810

[9] T. Fülöp and S. D. Katz, A frame and gauge free formulation of quantum mechanics. auant-ph/9806067 (1998).

[10] L. Hsiao and T.-P. Liu, Nonlinear diffusive phenomena of nonlinear hyperbolic systems, *Chin. Ann. Math.* **14B** (1993), 465-480.

[11] F. Huang, H.-L. Li and A. Matsumura, Existence and stability of steady-state of one-dimensional quantum Euler-Poisson system for semiconductors, preprint 2003.

[12] A.Jüngel, *Quasi-hydrodynamic semiconductor equations*, Progress in Nonlinear Differential Equations, Birkhäuser, Basel 2001.

[13] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, *Lecture Notes in Math.*, **448** (1975), 25–70, Springer, Berlin.

[14] K. K. Kan and J. J. Griffin, Quantized friction and the correspondence principle: single particle with friction, *Physics Letters.* **50B(2)**: (1974) 241-243.

[15] M. D. Kostin, On the Schrödinger–Langevin equations, *J. Chem. Phys.* **57**, (1972) 2589–3591.

[16] H.-L. Li and C.-K. Lin, Semiclassical limit and well-posedness of Schrödinger-Poisson system, *Electronic Journal of Differential Equations*, Vol. 2003(2003), No. 93, 1-17

[17] H.-L. Li and K. Saxton, Asymptotic behavior of solutions to quasilinear hyperbolic equations with nonlinear damping, *Quarterly Appl. Math.* **61** (2003), no. 2, 295–313.

[18] E. Madelung, Quantentheorie in hydrodynamischer Form, *Z. Physik*, **40** (1927), 322.

[19] P.A. Markowich, C. Ringhofer and Schmeiser, C. *Semiconductor Equations*, Springer-Verlag 1989.

[20] M.B. Mensky, Continuous quantum measurements and particle integrals, Institute of Physica Publishing, Bristol and Philalphia, 1993.

[21] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, *J. Differential Equations* **131** (1996), 171–188.

[22] K. Nishihara, W. Wang, and T. Yang, L_p-convergence rate to nonlinear diffusion waves for p-system with damping, *J. Differential Equations* **161** (2000), 191–218.

[23] H. Risken, *The Fokker-Planck equation, methods of solution and application*, Springer-Verlag Berlin 1988.

[24] T.P. Spiller, P.S. Spencer, T.D. Clark, J.F. Ralph, H. Prace, R.J. Prance and A. Clippingdale, Adding dissipation to the Schrödinger equation from the quantum-potential viewpoint, *Foundations of Physics Letters* **4** (1991), 507–521.

[25] U. Weiss, *Quantum dissipative systems*, World Scientific, Singapore 1993.

[26] R. J. Wysocki, Quantum equations of motion for a dissipative system. *Physical Review A*, 61 (022104) (2000).

[27] K. Yasue, Stochastic quantization: A review. *International Journal of Theoretical Physics,* **18(12)** (1979), 861-913.