# Free Boundary Problems and Perpetual American Strangles

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**Abstract**

We consider the perpetual American strangles in the geometric jump-diffusion models. We assume further that the jump distribution is a mixture of exponential distributions. To solve the corresponding optimal stopping problem for this option, by using the approach in [5], we derive a system of equations that is equivalent to the associated free boundary problem with smooth pasting condition. We verify the existence of the solutions to these equations. Then, in terms of the solutions together with a verification theorem, we solve the optimal stopping problem and hence find the optimal exercise boundaries and the rational price for the perpetual American strangle. In addition we work out an algorithm for computing the optimal exercise boundaries and the rational price of this option.

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## 1 Introduction

An American option is an option that can be exercised at any time prior to its expiration time. For an American call option with a finite expiration time, Merton [11] observed that the price of the American option(written on an underlying stock without dividends) coincides with the price of the corresponding European option. However the American put option(even without dividends) presents a difficult problem. We have no explicit pricing formulas and the optimal exercise boundaries are not known. One exception is the perpetual American put option, i.e., an American put with infinite expiration time. Within the Black-Scholes model, the perpetual American put problem was solved by Mckean [10]. In the Lévy-based models, using the theory of pseudo-differential operators, Boyarchenko and Levendorskii [4] derived a closed formula for prices of perpetual American put and call options. By probabilistic techniques, Mordecki and Salminen [12] obtained explicit formulas under the assumption of mixed-exponentially distributed and arbitrary negative jumps for the call options, and negative mixed-exponentially distributed and arbitrary positive jumps for put options. (For related works, see Asmussen et al. [1] and the references therein.) In this paper we consider the pricing problem for the perpetual American strangles, which is a combination of a put and a call written on the same security.

Mathematically the pricing problem for perpetual American contracts in the Lévy-based model is equivalent to the optimal stopping problem of the form

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) \tag{1.1}$$

where $X = \{X_t : t \geq 0\}$ under the chosen risk-neutral probability measure $\mathbb{P}_x$ is a Lévy process started from $X_0 = x$. Further, $g$ is the nonnegative continuous reward function corresponding to the

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contract, $r \geq 0$ and $\mathcal{T}$ is a family of stopping times with respect to the natural filtration generated by $X$, $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$. (Here we define, on $\tau = \infty$, $e^{-r\tau} g(X_\tau) = 0$.) In the literature, there are many approaches for finding the value functions $V(x)$ and the optimal stopping times $\tau^*$ such that $V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$. The free boundary approach is based on the observation that, under some suitable conditions, the value function $V(x)$ for the optimal stopping problem (1.1) is a solution to the free boundary(or Stephan) problem

$$(\mathcal{L}_X - r)V(x) = 0 \quad \text{in } \mathcal{C} \tag{1.2}$$

$$V(x) = g(x) \quad \text{on } \mathcal{D} \tag{1.3}$$

where $\mathcal{C} = \{x \in \mathbb{R}, V(x) > g(x)\}$ (the continuation region) , $\mathcal{D} = \{x \in \mathbb{R}, V(x) = g(x)\}$ (the stopping region) and $\mathcal{L}_X$ is the infinitesimal operator of X. (For details, see Shiryayev[17] Theorem 15 p.157.) Many authors in the literature also observed that the boundary of the stopping region $\mathcal{D}$ is determined by imposing the smooth pasting condition for the value function. Then, to solve the optimal stopping problem (1.1), it suffices to solve the above free boundary problem with suitable pasting conditions and prove a verification theorem(a verification theorem implies that solving the free boundary problem with smooth pasting condition(or related conditions) allow one to establish explicit solutions of the optimal stopping problem). By this approach, we find the value functions and the optimal stopping times for the optimal stopping problems (1.1) and, by the risk-neutral pricing formula, we obtain the optimal exercise times and the rational prices for the perpetual American contracts. For recent works and other approaches, see Kyprianou and Surya [9], Mordecki and Salminen [12], Mordecki [13], Novikov and Shiryaev [14], Surya [18] and the monograph of Peskir and Shiryaev [15].

It is worth noting that the reward functions considered above are of American put-type or American call-type. For our financial applications, we need to consider the reward functions $g$ of the two-sided form as in (2.4). (In the literature there are not many works for two-sided reward functions. See, for example, Beibel and Lerche [2], Gapeev and Lerche [7] and Boyarchenko [3].) Then we study the perpetual American strangles in the geometric jump-diffusion models. We assume further that the jump distribution is a two-sided mixture of exponential distributions. To solve the corresponding optimal stopping problem for this option, by using the approach in [5], we first derive a system of equations(see (3.3)-(3.8) below) that is equivalent to the associated free boundary problem with smooth pasting condition. In terms of the solutions to the system of equations , together with a verification theorem(Theorem 2.1), we find the optimal stopping time and the value function for the optimal stopping problem(Theorem 3.1). Hence, by the risk-neutral pricing formula, we obtain the optimal exercise boundaries and the price for the perpetual American strangle. In addition, in the proof of the existence of solutions to the equations (3.3)-(3.8), we also work out an algorithm for computing the optimal exercise boundaries and the rational price of the option(see Theorem 4.1 ).

The paper is organized as follows. In Section 2 we introduce our jump-diffusion setting and provide a verification theorem for the optimal stopping problems (1.1) with a general two-sided reward function. In Section 3 we consider the perpetual American strangles under the geometric jump-diffusion setting. We derive a system of equations for solving the corresponding free boundary problem with smooth pasting condition and in terms of solutions to the system of equations, we solve the optimal stopping problem corresponding to the perpetual American strangle. In Section 4 we prove the existence of solutions to the system of equations(Theorem 4.1). Some numerical results based on our algorithm are presented in Section 5. Section 6 concludes this paper. Long and difficult proofs are relegated to the Appendix.

## 2 Optimal Stopping and Jump-Diffusion Processes

Throughout this paper, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a jump-diffusion process $X$ of the form

$$X_t = ct + \sigma B_t + \sum_{n=1}^{N_t} Y_n \tag{2.1}$$

where $c \in \mathbb{R}$, $\sigma > 0$, $B = (B_t, t \geq 0)$ is a standard Brownian motion, $(N_t; t \geq 0)$ is a Poisson process with rate $\lambda > 0$. Also, $Y = (Y_n, n \geq 0)$ is a sequence of independent random variables with identical piecewise continuous density functions $f$. Assume further that $B, N_t$ and $Y$ are independent. A jump-diffusion process starting from $x$ is simply defined as $x + X_t$ for $t \geq 0$ and we denote its law by $\mathbb{P}_x$. For convenience we shall write $\mathbb{P}$ in place of $\mathbb{P}_0$. Also $\mathbb{E}_x$ denotes the expectation with respect to the probability measure $\mathbb{P}_x$. Under these model assumptions, we have $\mathbb{E}(e^{zX_t}) = e^{t\psi(z)}, z \in i\mathbb{R}$, where $\psi$ is called the characteristic exponent $\psi$ of $X$ and is given by the formula

$$\psi(z) = \frac{\sigma^2}{2}u^2 + cz + \lambda \int e^{zy} f(y)dy - \lambda. \tag{2.2}$$

Also the infinitesimal generator $\mathcal{L}_X$ of $X$ has a domain containing $C_0^2(\mathbb{R})$ and for $h \in C_0^2(\mathbb{R})$,

$$\mathcal{L}_X h(x) = \frac{1}{2}\sigma^2 h''(x) + ch'(x) + \lambda \int h(x+y)f(y)dy - \lambda h(x). \tag{2.3}$$

We define $\mathcal{L}_X h(x)$ by the expression (2.3) for all functions $h$ on $\mathbb{R}$ such that $h'$, $h''$ and the integral in (2.3) exists at $x$.

Given a jump-diffusion process $X$ as in (2.1), we consider in this section the optimal stopping problem (1.1) with the continuous reward function $g$ given by the formula

$$g(x) = g_1(x)\mathbf{1}_{\{x \leq l_1\}} + g_2(x)\mathbf{1}_{\{x \geq l_2\}} \tag{2.4}$$

for some $-\infty < l_1 \leq l_2 < \infty$. Here $g_1(x)$ is a strictly positive $C^\infty$-function on $(-\infty, l_1)$ and $g_2(x)$ is a strictly positive $C^\infty$-function on $(l_2, \infty)$. We assume further that $\mathbb{E}_x\left[\sup_{t \geq 0} e^{-rt} g(X_t)\right] < \infty$ for all $x \in \mathbb{R}$. For any set $I$ in $\mathbb{R}$, we write $\tau_I = \inf\{t \geq 0 | X_t \in I\}$ and set

$$V_I(x) = \mathbb{E}_x[e^{-r\tau_I} g(X_{\tau_I})], \qquad x \in \mathbb{R}. \tag{2.5}$$

**Theorem 2.1** *Given $I = (h_1, h_2)^c$ where $-\infty < h_1 < l_1 \leq l_2 < h_2 < \infty$. Assume that the function $V_I(x)$ in (2.5) satisfies the following conditions:*

*(a) $V_I(x)$ is the difference of two convex functions.*

*(b) $V_I(x)$ is a twice continuously differentiable function except possibly at $h_1$ and $h_2$.*

*(c) The limits $V_I''(h_i\pm) = \lim_{h \to h_i \pm} V_I''(h)$, $i = 1, 2$, exist and are finite.*

*(d) $(\mathcal{L}_X - r)V_I(x) \leq 0$ for all $x$ except possibly at $h_1$ and $h_2$.*

*(e) $V_I(x) \geq g(x)$ for all $x \in (h_1, h_2)$.*

*Then $V_I(x)$ is the value function for the optimal stopping problem (1.1) with the reward function $g$ given in (2.4).*

**Proof :** Let $V$ be the value function for the optimal stopping problem (1.1). Clearly, we have $V_I(x) \leq V(x)$. It remains to show that $V(x) \leq V_I(x)$. By the Meyer-Itô formula(see, for example, Corollary 1 in Protter [16] ChIV. pp.218-pp.219), we have

$$\begin{aligned} e^{-rt}V_I(X_t) - V_I(x) &= -\int_0^t re^{-rs}V_I(X_s)ds + \int_0^t e^{-rs}V_I'(X_{s-})dX_s \\ &+ \sum_{0<s\leq t} e^{-rs}\left(V_I(X_s) - V_I(X_{s-}) - V_I'(X_{s-})\Delta X_s\right) + \frac{1}{2}\int_0^t e^{-rs}V_I''(X_{s-})d[X,X]_s^c \end{aligned}$$

where $V_I'(x)$ is its left derivative and $V_I''(x)$ is the second derivative in the generalized function sense. By similar arguments as that in Mordecki [13] Sec. 3, we have

$$e^{-rt}V_I(X_t) - V_I(x) = \int_0^t e^{-rs}(\mathcal{L}_X - r)V_I(X_{s-})ds + M_t \tag{2.6}$$

where $\{M_t\}$ is a local martingale with $M_0 = 0$. Let $T_n \uparrow \infty$ be a sequence of stopping times such that for each $n$, $\{M_{T_n \wedge t}\}$ is a martingale. Let $\tau$ be a stopping time. By the optional stopping theorem, we have $\mathbb{E}_x[M_{T_n \wedge t \wedge \tau}] = \mathbb{E}_x[M_0] = 0$. In addition, by (d), we have $\int_0^{T_n \wedge t \wedge \tau} e^{-rs}(\mathcal{L}_X -$

$r)V_I(X_{s-})ds \le 0$. By (2.6), we observe $\mathbb{E}_x[e^{-r(T_n\wedge t\wedge\tau)}V_I(X_{T_n\wedge t\wedge\tau})] \le V_I(x)$. Since $g(x) \ge 0$ and $\mathbb{E}_x\left[\sup_{t\ge 0} e^{-rt}g(X_t)\right] < \infty$, by Dominated Convergence Theorem and (e), we have

$$\mathbb{E}_x[e^{-r\tau}g(X_\tau)] = \mathbb{E}_x[\lim_{t\to\infty}\lim_{n\to\infty} e^{-r(\tau\wedge t\wedge T_n)}g(X_{(\tau\wedge t\wedge T_n)})] = \lim_{t\to\infty}\lim_{n\to\infty}\mathbb{E}_x[e^{-r(\tau\wedge t\wedge T_n)}g(X_{(\tau\wedge t\wedge T_n)})]$$
$$\le \lim_{t\to\infty}\lim_{n\to\infty}\mathbb{E}_x[e^{-r(\tau\wedge t\wedge T_n)}V_I(X_{(\tau\wedge t\wedge T_n)})] \le V_I(x).$$

Because $\tau$ is arbitrary, we observe $V(x) = \sup_\tau \mathbb{E}_x[e^{-r\tau}g(X_\tau)] \le V_I(x)$. The proof is complete. $\square$

We have the uniqueness of solutions for the boundary value problem in (1.2) and (1.3).

**Proposition 2.1** *Assume that $g_1$ is bounded on $(-\infty, l_1)$ and the function $\int_0^\infty g_2(x+y)f(y)dy, x \ge l_2$, is locally bounded. Let $I = (h_1, h_2)^c$ for some $-\infty < h_1 < l_1 \le l_2 < h_2 < \infty$. If $\widetilde{V}$ is a solution of the boundary value problem:*

$$\begin{cases} (\mathcal{L}_X - r)\widetilde{V}(x) = 0, & \forall x \in (h_1, h_2) \\ \widetilde{V}(x) = g(x), & \forall x \in I \end{cases} \tag{2.7}$$

*and $\widetilde{V}$ is in $C^2(h_1, h_2) \cap C[h_1, h_2]$, then $\widetilde{V}(x) = V_I(x)$ for all $x \in \mathbb{R}$.*

**Proof:** See the Appendix. $\square$

**Remark.** The conclusion of Proposition 2.1 still holds if the functions $g_1$ and $g_2$ are $C^\infty$(not necessary strictly positive) and satisfy the conditions in Proposition 2.1. $\square$

**Proposition 2.2** *Assume that $g_1$ and $g_1'$ are bounded on $(-\infty, l_1)$ and the functions $\int_0^\infty g_2(x+y)f(y)dy$ and $\int_0^\infty g_2'(x+y)f(y)dy, x \ge l_2$, are locally bounded. We assume further that $g_1(x) - g_1'(x)$ is positive and increasing on $(-\infty, l_1)$, $g_2(x) - g_2'(x)$ is negative and decreasing on $(l_2, \infty)$ and $\mathbb{E}_x[\sup_{t\ge 0} e^{-rt}|g'(X_t)|] < \infty$ for all $x$. Let $I = (h_1, h_2)^c$ for some $-\infty < h_1 < l_1 \le l_2 < h_2 < \infty$ and consider a non-negative function $\widetilde{V}(x)$ on $\mathbb{R}$ that is $C^2$ on $(h_1, h_2)$ and satisfies the following conditions:*

*(a)* $(\mathcal{L}_X - r)\widetilde{V}(x) = 0, \forall x \in (h_1, h_2)$,

*(b)* $\widetilde{V}(x) = g(x), \forall x \in I$.

*(c)* $\frac{d}{dx}\int \widetilde{V}(x+y)f(y)dy = \int \widetilde{V}'(x+y)f(y)dy, \forall x \in (h_1, h_2)$.

*(d) $\widetilde{V}$ is continuous at $h_1$ and $h_2$ and $\widetilde{V}'(h_i), i = 1, 2$, exist and are continuous there.*

*Then $\widetilde{V}(x) \ge g(x)$ for all $x \in (h_1, h_2)$.*

**Proof:** By Proposition 2.1, we have $\widetilde{V}(x) = V_I(x)$ for all $x \in \mathbb{R}$. Note that $\widetilde{V}$ is $C^\infty$ on $(h_1, h_2)$ (for a proof, see Chen et al. [6]) and, for $x \in (h_1, h_2)$, we have

$$0 = \frac{d}{dx}(\mathcal{L}_X - r)\widetilde{V}(x) = \frac{1}{2}\sigma^2\widetilde{V}'''(x) + c\widetilde{V}''(x) - (\lambda + r)\widetilde{V}'(x) + \lambda\int \widetilde{V}'(x+y)f(y)dy,$$

which implies that $(\mathcal{L}_X - r)\widetilde{V}'(x) = 0$ for $x \in (h_1, h_2)$. By condition (d), $\widetilde{V}' \in C[h_1, h_2]$ and hence by the remark after Proposition 2.1, $\widetilde{V}'(x) = E_x[e^{-r\tau_I}g'(X_{\tau_I})]$. This implies that $\widetilde{V}(x)$ satisfies the ODE: $\widetilde{V}'(x) - \widetilde{V}(x) = F(x)$, where $F(x) = \mathbb{E}_x[e^{-r\tau_I}(g'(X_{\tau_I}) - g(X_{\tau_I}))]$. First consider the case that $h_1 \le x \le l_1$. By the ODE theory and the boundary conditions, we have $\widetilde{V}(x) = e^x\left(\int_{h_1}^x e^{-t}F(t)dt + g_1(h_1)e^{-h_1}\right)$. Set $H(x) \equiv e^{-x}(\widetilde{V}(x) - g(x))$. Then $H(x) = \int_{h_1}^x e^{-t}F(t)dt +$

$g_1(h_1)e^{-h_1} - g_1(x)e^{-x}$ and

$$
\begin{aligned}
H'(x) \quad & = e^{-x}F(x) + g_1(x)e^{-x} - g_1'(x)e^{-x} \\
& = e^{-x}\{\mathbb{E}_x[e^{-r\tau_I}(g'(X_{\tau_I}) - g(X_{\tau_I}))] + g_1(x) - g_1'(x)\} \\
& = e^{-x}\{\mathbb{E}_x[e^{-r\tau_I^+}(g_2'(X_{\tau_I}) - g_2(X_{\tau_I})); \{\tau_I = \tau_I^+\}] \\
& +\mathbb{E}_x[e^{-r\tau_I^-}(g_1'(X_{\tau_I}) - g_1(X_{\tau_I})); \{\tau_I = \tau_I^-\}] + g_1(x) - g_1'(x)\} \\
& \geq e^{-x}\mathbb{E}_x[e^{-r\tau_I^+}(g_2'(X_{\tau_I}) - g_2(X_{\tau_I})); \{\tau_I = \tau_I^+\}] \\
& +e^{-x}(g_1(x) - g_1'(x))(1 - \mathbb{E}_x[e^{-r\tau_I^-}; \{\tau_I = \tau_I^-\}]
\end{aligned}
$$

where $\tau_I^+ = \inf\{t \geq 0 | X_t \geq h_2\}$ and $\tau_I^- = \inf\{t \geq 0 | X_t \leq h_1\}$. For the last inequality, we use the facts that $g_1(x) - g_1'(x)$ is increasing and hence $g_1(X_{\tau_I^-}) - g_1'(X_{\tau_I^-}) \leq g_1(h_1) - g_1'(h_1) \leq g_1(x) - g_1'(x)$. Since $g_2(x) - g_2'(x)$ is negative and $g_1(x) - g_1'(x)$ is positive, we obtain $H'(x) \geq 0$ which implies that $H(x)$ is increasing . Therefore $H(x) \geq H(h_1) = 0$ and hence $\widetilde{V}(x) \geq g(x)$. By a similar argument, we get $\widetilde{V}(x) \geq g(x)$ for $l_2 \leq x \leq h_2$. Since $\widetilde{V}(x) = V_I(x) \geq 0 = g(x)$ for $l_1 \leq x \leq l_2$, we complete the proof. □

## 3 Perpetual American Strangles and Straddles

A strangle is a financial instrument whose payoff function is a combination of a put with the strike price $K_1$ and a call with the strike price $K_2$ written on the same security, where $K_1 \leq K_2$. In particular, if $K_1 = K_2$, the strangle becomes a straddle. We model the price of the underlying security under the chosen risk-neutral measure by a geometric jump-diffusion: $S_t = \exp\{X_t\}$. Here $X$ is a jump-diffusion process of the form in (2.1). We assume further that the jump density function $f$ is given by the mixture of exponential distributions

$$
f(x) = \sum_{i=1}^{N^+} p_i \eta_i^+ e^{-\eta_i^+ x} 1_{\{x>0\}} + \sum_{j=1}^{N^-} q_j(-\eta_j^-)e^{-\eta_j^- x} 1_{\{x<0\}} \tag{3.1}
$$

where $\eta_1^- < ... < \eta_{N^-}^- < 0 < \eta_1^+ < ... < \eta_{N^+}^+$, $p_i$'s and $q_j$'s are positive with $\sum_{i=1} p_i + \sum_{j=1} q_j = 1$. (In a Lévy model, there are infinitely many equivalent risk-neutral measures and, for pricing purpose, we usually choose one of them by using the so-called Cramér-Esscher transform. Note that this transform preserves the jump-diffusion structure as above. For details, see in particular Appendix A of Asmussen et al. [1].) The characteristic exponent for this jump-diffusion process $X$ is given by the formula

$$
\psi(z) = \frac{1}{2}\sigma^2 z^2 + cz + \lambda(\sum_{i=1}^{N^+} \frac{p_i \eta_i^+}{\eta_i^+ - z} + \sum_{j=1}^{N^-} \frac{q_j \eta_j^-}{\eta_j^- - z}) - \lambda.
$$

The rational price for the perpetual American strangle is the value function for the optimal stopping problem (1.1) with the reward function $g$ given by the formula

$$
g(x) = (K_1 - e^x)^+ + (e^x - K_2)^+ = g_1(x)1_{x \leq l_1} + g_2(x)1_{x \geq l_2}. \tag{3.2}
$$

where $l_1 = \ln K_1, l_2 = \ln K_2, g_1(x) = K_1 - e^x$ and $g_2(x) = e^x - K_2$. To find the value function, we consider the interval $(h_1, h_2)$ with $h_1 < l_1 \leq l_2 < h_2$. First we find the function $V(x)$ that solves the boundary value problem (2.7). As in Chen et al. [5], we first transform the integro-differential equation in (2.7) into the ODE

$$
\prod_{i=1}^{N^+}(\eta_i^+ - D)\prod_{j=1}^{N^-}(\eta_j^- - D)(\frac{1}{2}\sigma^2 D^2 + cD - (\lambda + r))V(x) +
$$

$$
\lambda \prod_{i=1}^{N^+}(\eta_i^+ - D)\prod_{j=1}^{N^-}(\eta_j^- - D)(\sum_{i=1}^{N^+} \frac{p_i \eta_i^+}{\eta_i^+ - D} + \sum_{j=1}^{N^-} \frac{q_j \eta_j^-}{\eta_j^- - D}))V(x) = 0
$$

where $D = \frac{d}{dx}$. By the general theory of ODE theory, the function $V$ in $(h_1, h_2)$ must be of the form $V(x) = \sum_{n=1}^{N^-+N^++2} C_n e^{\beta_n x}$, where $\beta_n$ are the roots to the characteristic polynomial $\phi(x)$ of the above ODE, that is,

$$\phi(x) = \prod_{i=1}^{N^+}(\eta_i^+ - x)\prod_{j=1}^{N^-}(\eta_j^- - x)\left[\frac{1}{2}\sigma^2 x^2 + cx - (\lambda + r) + \lambda(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - x} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - x})\right]$$

(Note that in fact we have $\beta_1 < \eta_1^- < \beta_2 < \eta_2^- < \cdots < \beta_{N^-} < \eta_{N^-}^- < \beta_{N^-+1} < 0 < \beta_{N^-+2} < \eta_1^+ < \cdots < \beta_{N^-+N^++1} < \eta_{N^+}^+ < \beta_{N^-+N^++2}$.) For $x \notin (h_1, h_2)$, we set $V(x) = g(x)$.

To determine the coefficients $C_n$'s, plugging the function $V$ into the integro-differential equation in (2.7), we obtain the system of equations

$$\sum_{n=1}^{N^++N^-+2} C_n \frac{e^{\beta_n h_2}}{\beta_n - \eta_k^+} = \frac{1}{1-\eta_k^+}e^{h_2} + \frac{K_2}{\eta_k^+}, \quad k = 1, 2, ..., N^+ \tag{3.3}$$

$$\sum_{n=1}^{N^++N^-+2} C_n \frac{e^{\beta_n h_1}}{\beta_n - \eta_k^-} = -\frac{1}{1-\eta_k^-}e^{h_1} - \frac{K_1}{\eta_k^-}, \quad k = 1, 2, ..., N^- \tag{3.4}$$

(For details, see ([5],[6]).) Also, imposing the condition (d) of Proposition 2.2 for the function $V(x)$ (i.e., assuming that $V$ satisfies the continuity and smooth pasting conditions at the boundaries) gives the equations

$$\sum_{n=1}^{N^++N^-+2} C_n e^{\beta_n h_2} = e^{h_2} - K_2 \tag{3.5}$$

$$\sum_{n=1}^{N^++N^-+2} C_n e^{\beta_n h_1} = K_1 - e^{h_1} \tag{3.6}$$

$$\sum_{n=1}^{N^++N^-+2} C_n \beta_n e^{\beta_n h_2} = e^{h_2} \tag{3.7}$$

$$\sum_{n=1}^{N^++N^-+2} C_n \beta_n e^{\beta_n h_1} = -e^{h_1} \tag{3.8}$$

Now, with the set $\{C_1, \cdots, C_{N^-+N^++2}, h_1, h_2\}$ that satisfies the equations (3.3)-(3.8), we will show later that the function $V$ is the value function for the optimal stopping problem (1.1). To do this, we need some further properties for the coefficients $C_n$'s. We consider the following conditions on the model :

$$\eta_i^+ > 1 \text{ for } i = 1, 2, ..., N^+ \tag{3.9}$$

and

$$\frac{1}{2}\sigma^2 + c - (\lambda + r) + \lambda\left(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - 1} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - 1}\right) < 0 \tag{3.10}$$

(Note that (3.9) implies that $\mathbb{E}[e^{X_1}] < \infty$ and (3.10) guarantees $\mathbb{E}[e^{X_1}] < e^r$( hence the underlying asset pays dividends continuously). If $\mathbb{E}[e^{X_1}] < e^r$ and $0 \le g(x) \le A + Be^x$ for some constants $A$ and $B$, then $\mathbb{E}[\sup_{t\ge0} e^{-rt}g(X_t)] < \infty$. For details, see Lemma 4.1 of Mordecki and Salminen [12].)

**Lemma 3.1** *Under the conditions (3.9) and (3.10), we have* $\beta_{N^-+2} > 1$.

**Proof :** See the Appendix. □

**Example 1.** Consider the case that $X = ct + \sigma B_t$ with $\frac{1}{2}\sigma^2 + c < r$. Since the process $X$ does not have the jump part, the system of equations (3.3)-(3.8) is reduced to the following

$$C_1 e^{\beta_1 h_2} + C_2 e^{\beta_2 h_2} = e^{h_2} - K_2 \tag{3.11}$$
$$C_1 e^{\beta_1 h_1} + C_2 e^{\beta_2 h_1} = K_1 - e^{h_1} \tag{3.12}$$
$$C_1 \beta_1 e^{\beta_1 h_2} + C_2 \beta_2 e^{\beta_2 h_2} = e^{h_2} \tag{3.13}$$
$$C_1 \beta_1 e^{\beta_1 h_1} + C_2 \beta_2 e^{\beta_2 h_1} = -e^{h_1} \tag{3.14}$$

where $\beta_1$ and $\beta_2$ are solutions to the equation $\frac{1}{2}\sigma^2 x^2 + cx - r = 0$, that is, $\beta_1 = \frac{-c-\sqrt{c^2+2r\sigma^2}}{\sigma^2}$ and $\beta_2 = \frac{-c+\sqrt{c^2+2r\sigma^2}}{\sigma^2}$. By (3.11) and (3.12), we obtain

$$C_1 = \frac{e^{\beta_2 h_1}(e^{h_2} - K_2) - e^{\beta_2 h_2}(K_1 - e^{h_1})}{detA} \quad \text{and} \quad C_2 = \frac{e^{\beta_1 h_2}(K_1 - e^{h_1}) - e^{\beta_1 h_1}(e^{h_2} - K_2)}{detA} \tag{3.15}$$

where $A = \begin{bmatrix} e^{\beta_1 h_2} & e^{\beta_2 h_2} \\ e^{\beta_1 h_1} & e^{\beta_2 h_1} \end{bmatrix}$. Hence, in terms of $h_1$ and $h_2$, we have explicit formulas for $C_1$ and $C_2$. To determine $h_1$ and $h_2$, plugging the expressions for $C_1$ and $C_2$ in (3.15) into (3.13) and (3.14), we observe that equations (3.13) and (3.14) are equivalent to the equations:

$$\frac{(K_1 - e^{h_1})\beta_2 e^{\beta_2 h_1} + e^{h_1} e^{\beta_2 h_1}}{\beta_2 e^{\beta_2 h_1} e^{\beta_1 h_1} - e^{\beta_2 h_1}\beta_1 e^{\beta_1 h_1}} = \frac{(e^{h_2} - K_2)\beta_2 e^{\beta_2 h_2} - e^{h_2} e^{\beta_2 h_2}}{\beta_2 e^{\beta_2 h_2} e^{\beta_1 h_2} - e^{\beta_2 h_2}\beta_1 e^{\beta_1 h_2}} \tag{3.16}$$
$$\frac{(K_1 - e^{h_1})\beta_1 e^{\beta_1 h_1} + e^{h_1} e^{\beta_1 h_1}}{\beta_2 e^{\beta_2 h_1} e^{\beta_1 h_1} - e^{\beta_2 h_1}\beta_1 e^{\beta_1 h_1}} = \frac{(e^{h_2} - K_2)\beta_1 e^{\beta_1 h_2} - e^{h_2} e^{\beta_1 h_2}}{\beta_2 e^{\beta_2 h_2} e^{\beta_1 h_2} - e^{\beta_2 h_2}\beta_1 e^{\beta_1 h_2}} \tag{3.17}$$

Gapeev and Lerche [7] showed that there is a unique solution $h_1^*, h_2^*$ to the equations (3.16) and (3.17). Then, by a verification lemma, they verified that $(h_1^*, h_2^*)$ is the continuation region for the corresponding optimal stopping problem and the value function on $(h_1^*, h_2^*)$ is given by the formula $V(x) = C_1^* e^{\beta_1 x} + C_2^* e^{\beta_2 x}$. Here $C_1^*$ and $C_2^*$ are computed by the formulas in (3.15) with $h_1, h_2$ replaced by $h_1^*$ and $h_2^*$. For a martingale approach for this optimal stopping problem, see Beibel and Lerche [2].

In the following, we solve the optimal stopping problem (1.1) by using the results in Section 2. Our approach also gives an algorithm for finding the solutions to the system of equations (3.11)-(3.14). (In fact our method will be applied later for processes with jumps.) Assume that $\{C_1, C_2, h_1, h_2\}$ is a solution to the equations (3.11)-(3.14). From these equations, we have $ADC = K$ where

$$D = \begin{bmatrix} \beta_1 - 1 & 0 \\ 0 & \beta_2 - 1 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} K_2 \\ -K_1 \end{bmatrix}.$$

From this, we have

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{detA} \begin{bmatrix} \frac{1}{\beta_1 - 1}(e^{\beta_2 h_2} K_1 + e^{\beta_2 h_1} K_2) \\ \frac{-1}{\beta_2 - 1}(e^{\beta_1 h_2} K_1 + e^{\beta_1 h_1} K_2) \end{bmatrix} \tag{3.18}$$

uniquely determined by $h_1$ and $h_2$. (By Lemma 3.1, $\beta_2 > 1$. Hence $C_1$ and $C_2$ have the same sign.) On the other hand, the matrix form of (3.13) and (3.14) is given by

$$\begin{bmatrix} \beta_1 e^{\beta_1 h_2} & \beta_2 e^{\beta_2 h_2} \\ \beta_1 e^{\beta_1 h_1} & \beta_2 e^{\beta_2 h_1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} e^{h_2} \\ -e^{h_1} \end{bmatrix} \tag{3.19}$$

By multiplying $e^{-h_2}$ to both sides of (3.13), $e^{-h_1}$ to both sides of (3.14) and adding them together, we have $C_1\beta_1(e^{(\beta_1-1)h_1} + e^{(\beta_1-1)h_2}) + C_2\beta_2(e^{(\beta_2-1)h_1} + e^{(\beta_2-1)h_2}) = 0$ Combining this with the expressions for $C_1$ and $C_2$ in (3.18) gives

$$det \begin{bmatrix} \beta_1(e^{(\beta_1-1)h_1} + e^{(\beta_1-1)h_2}) & \beta_2(e^{(\beta_2-1)h_1} + e^{(\beta_2-1)h_2}) \\ \frac{1}{\beta_2-1}(e^{\beta_1 h_2} K_1 + e^{\beta_1 h_1} K_2) & \frac{1}{\beta_1-1}(e^{\beta_2 h_2} K_1 + e^{\beta_2 h_1} K_2) \end{bmatrix} = 0 \tag{3.20}$$

By multiplying $e^{-\beta_1 h_1}$ to the first column of the matrix in (3.20), $e^{-\beta_2 h_1}$ to the second column and then multiplying $\frac{e^{h_1}}{\beta_1\beta_2}$ to the first row and $\frac{1}{K_2}$ to the second row, we obtain

$$det \begin{bmatrix} \frac{1}{\beta_2}(1 + e^{(\beta_1-1)\Delta h}) & \frac{1}{\beta_1}(1 + e^{(\beta_2-1)\Delta h}) \\ \frac{1}{\beta_2-1}(1 + \frac{K_1}{K_2} e^{\beta_1 \Delta h}) & \frac{1}{\beta_1-1}(1 + \frac{K_1}{K_2} e^{\beta_2 \Delta h}) \end{bmatrix} = 0 \tag{3.21}$$

where $\Delta h = h_2 - h_1$. In addition, from (3.14) and (3.18), we have

$$-e^{h_1} = \frac{1}{detA} det \begin{bmatrix} \frac{\beta_1}{\beta_1-1} e^{\beta_1 h_1} & \frac{-\beta_2}{\beta_2-1} e^{\beta_2 h_1} \\ K_1 e^{\beta_1 h_1} + K_2 e^{\beta_1 h_2} & K_1 e^{\beta_2 h_2} + K_2 e^{\beta_2 h_1} \end{bmatrix}$$

$$= \frac{det \begin{bmatrix} \frac{\beta_1}{\beta_1-1} e^{\beta_1 h_1} & \frac{-\beta_2}{\beta_2-1} e^{\beta_2 h_1} \\ K_1 e^{\beta_1 h_1} + K_2 e^{\beta_1 h_2} & K_1 e^{\beta_2 h_2} + K_2 e^{\beta_2 h_1} \end{bmatrix}}{det \begin{bmatrix} e^{\beta_1 h_2} & e^{\beta_2 h_2} \\ e^{\beta_1 h_1} & e^{\beta_2 h_1} \end{bmatrix}} = \frac{det \begin{bmatrix} \frac{\beta_1}{\beta_1-1} & \frac{-\beta_2}{\beta_2-1} \\ K_1 + K_2 e^{\beta_1 \Delta h} & K_1 e^{\beta_2 \Delta h} + K_2 \end{bmatrix}}{det \begin{bmatrix} e^{\beta_1 \Delta h} & e^{\beta_2 \Delta h} \\ 1 & 1 \end{bmatrix}}$$

which implies

$$h_1 = \log \left( - \frac{det \begin{bmatrix} \frac{\beta_1}{\beta_1-1} & \frac{-\beta_2}{\beta_2-1} \\ K_1 + K_2 e^{\beta_1 \Delta h} & K_1 e^{\beta_2 \Delta h} + K_2 \end{bmatrix}}{det \begin{bmatrix} e^{\beta_1 \Delta h} & e^{\beta_2 \Delta h} \\ 1 & 1 \end{bmatrix}} \right). \tag{3.22}$$

and hence, we also obtain $h_2 = h_1 + \Delta h$. With this $h_1$ and $h_2$, we compute $C_1, C_2$ by (3.18).

To prove the existence of solutions to the equations (3.11)-(3.14), we show that there is a solution $\widehat{\Delta h}$ to (3.21) in $(0, \infty)$. As $\Delta h \to 0$, the left term in (3.21) tends to

$$2\left(1 + \frac{K_1}{K_2}\right)\left(\frac{1}{\beta_2(\beta_1-1)} - \frac{1}{\beta_1(\beta_2-1)}\right) = \left(1 + \frac{K_1}{K_2}\right) \frac{2(\beta_2 - \beta_1)}{\beta_1\beta_2(\beta_1-1)(\beta_2-1)} > 0.$$

Since

$$det \begin{bmatrix} \frac{1+e^{(\beta_1-1)\Delta h}}{\beta_1-1} & \frac{1+e^{(\beta_2-1)\Delta h}}{\beta_2-1} \\ \frac{1+\frac{K_1}{K_2}e^{\beta_1\Delta h}}{\beta_1} & \frac{1+\frac{K_1}{K_2}e^{\beta_2\Delta h}}{\beta_2} \end{bmatrix} = e^{\beta_2 \Delta h} det \begin{bmatrix} \frac{1+e^{(\beta_1-1)\Delta h}}{\beta_1-1} & \frac{e^{-\beta_2\Delta h}+e^{-\Delta h}}{\beta_2-1} \\ \frac{1+\frac{K_1}{K_2}e^{\beta_1\Delta h}}{\beta_1} & \frac{e^{-\beta_2\Delta h}+\frac{K_1}{K_2}}{\beta_2} \end{bmatrix}, \tag{3.23}$$

we have, as $\Delta h \to \infty$,

$$det \begin{bmatrix} \frac{1+e^{(\beta_1-1)\Delta h}}{\beta_1-1} & \frac{e^{-\beta_2\Delta h}+e^{-\Delta h}}{\beta_2-1} \\ \frac{1+\frac{K_1}{K_2}e^{\beta_1\Delta h}}{\beta_1} & \frac{e^{-\beta_2\Delta h}+\frac{K_1}{K_2}}{\beta_2} \end{bmatrix} \to det \begin{bmatrix} \frac{1}{\beta_1-1} & 0 \\ \frac{1}{\beta_1} & \frac{K_1}{K_2\beta_2} \end{bmatrix} = \frac{K_1}{K_2(\beta_1-1)\beta_2} < 0.$$

Therefore, (3.21) has a solution $\widehat{\Delta h}$ in $(0, \infty)$ by the intermediate value theorem. With this $\widehat{\Delta h}$, we compute $\{C_1, C_2, h_1, h_2\}$ by the formulas (3.22), (3.18) and $h_2 = h_1 + \widehat{\Delta h}$. (Later in this paper, we show that $\{C_1, C_2, h_1, h_2\}$ is a solution to the system of equations (3.11)-(3.14).) Define the function $V(x)$ by the formula

$$V(x) = \begin{cases} C_1 e^{\beta_1 x} + C_2 e^{\beta_2 x} & \text{if} \quad x \in (h_1, h_2) \\ g(x) & \text{if} \quad x \in (h_1, h_2)^c \end{cases}$$

where $g$ is the function in (3.2). Then $V$ is a solution of the boundary value problem in Proposition 2.1. Hence we have $V(x) = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ for all $x \in \mathbb{R}$. Also, by Proposition 2.2 , we observe $V(x) \geq g(x)$ for all $x$. To prove that $V$ is indeed the value function, by Theorem 2.1, it remains to verify that $(\mathcal{L}_X - r)V(x) \leq 0$ for all $x \notin (h_1, h_2)$. For $x > h_2$, we have $(\mathcal{L}_X - r)V(x) = \frac{\sigma^2}{2} g''(x) + cg'(x) - rg(x) = (\frac{1}{2}\sigma^2 + c - r)e^x + rK_2$. Since $\frac{1}{2}\sigma^2 + c < r$, we observe $\frac{d}{dx}(\mathcal{L}_X - r)V(x) = (\frac{1}{2}\sigma^2 + c - r)e^x < 0$ which implies that $(\mathcal{L}_X - r)V(x)$ is a decreasing function on $(h_2, \infty)$. In addition, we have $(\mathcal{L}_X - r)V(x) = 0$ for $x \in (h_1, h_2)$ and, hence,

$$\begin{aligned} &(\mathcal{L}_X - r)V(h_2^+) = (\mathcal{L}_X - r)V(h_2^+) - (\mathcal{L}_X - r)V(h_2^-) \\ &= [\frac{1}{2}\sigma^2 V''(h_2^+) + cV'(h_2^+) - rV(h_2^+)] - [\frac{1}{2}\sigma^2 V''(h_2^-) + cV'(h_2^-) - rV(h_2^-)] \\ &= \frac{1}{2}\sigma^2 [V''(h_2^+) - V''(h_2^-)] = \frac{1}{2}\sigma^2 (e^{h_2} - \sum_{n=1}^{2} C_n \beta_n^2 e^{\beta_n h_2}) \end{aligned}$$

(The third equality holds since the function $V(x)$ satisfies both the continuous fit and the smooth pasting conditions at $h_2$.) Since $V(x) \geq 0$ and $C_1$ and $C_2$ have the same sign , we observe $C_i \geq 0$

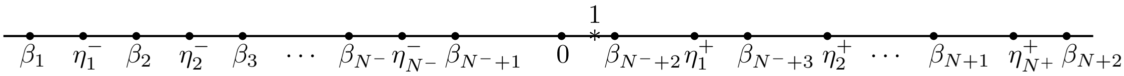


Figure 1: Relationship of $\eta_i^-$, $1 \le i \le N^-$, $\eta_j^+$, $1 \le j \le N^+$ and $\beta_n$, $1 \le n \le N+2$.

for $i = 1, 2$. Also we have $\beta_1 < 0 < 1 < \beta_2$. Therefore we observe $(\mathcal{L}_X - r)V(h_2^+) = \frac{1}{2}\sigma^2(e^{h_2} - \sum_{n=1}^{2} C_n\beta_n^2 e^{\beta_n h_2}) \le \frac{1}{2}\sigma^2(e^{h_2} - \sum_{n=1}^{2} C_n\beta_n e^{\beta_n h_2}) = 0$. This implies that $(\mathcal{L}_X - r)V(x) \le \mathcal{L}_X - r)V(h_2^+) \le 0$ for all $x > h_2$. By a similar argument, $(\mathcal{L}_X - r)V(x) \le 0$ for $x < h_1$. The proof is complete.

□

Now we go back to the equations (3.3)-(3.8). From this point on, we set $N = N^- + N^+$ and assume that the conditions (3.9) and (3.10) hold. Subtract (3.5) from (3.7) and (3.6) from (3.8), we have

$$\sum_{n=1}^{N+2} C_n(1-\beta_n)e^{\beta_n h_2} = -K_2 \tag{3.24}$$

$$\sum_{n=1}^{N+2} C_n(1-\beta_n)e^{\beta_n h_1} = K_1 \tag{3.25}$$

Using (3.25), (3.8) and (3.4), we have

$$\sum_{n=1}^{N+2} C_n \frac{\beta_n(1-\beta_n)}{\beta_n - \eta_k^-} e^{\beta_n h_1} = 0, \tag{3.26}$$

for $k = 1, 2, ..., N^-$. Similarly, by (3.24), (3.7) and (3.3), we have

$$\sum_{n=1}^{N+2} C_n \frac{\beta_n(1-\beta_n)}{\beta_n - \eta_k^+} e^{\beta_n h_2} = 0 \tag{3.27}$$

for $k = 1, 2, ..., N^+$. From equations (3.24) and (3.25), we also have

$$\sum_{i=1}^{N+2} C_n(1-\beta_n)(\frac{1}{K_1}e^{\beta_n h_1} + \frac{1}{K_2}e^{\beta_n h_2}) = 0. \tag{3.28}$$

In addition, by (3.7) and (3.8), we have

$$\sum_{i=1}^{N+2} C_n\beta_n(e^{(\beta_n-1)h_1} + e^{(\beta_n-1)h_2}) = 0. \tag{3.29}$$

**Lemma 3.2** *Assume that $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ is a solution of the equations (3.3)-(3.8). Then $C_j \ne 0$ except for at most one $j$.*

**Proof :** See the Appendix. □

**Lemma 3.3** *Assume that $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ is a solution of the equations (3.3)-(3.8). Then $C_n \ge 0$ for all $n$.*

**Proof :** See the Appexdix. □

**Lemma 3.4** *Assume that $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x_0} = e^{x_0}$ for some $x_0 \in \mathbb{R}$. Then there exists $\epsilon > 0$ such that $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} < e^x$ for all $x \in (x_0 - \epsilon, x_0)$ . Also we have $\sum_{n=1}^{N+2} C_n\beta_n e^{\beta_n x} \ge e^x$ for all $x \ge x_0$.*

**Proof:** See the Appendix. □

**Theorem 3.1** *Let $\{C_1, ..., C_N, h_1, h_2\}$ be a solution of the equations (3.3)-(3.8). Define the function $V(x)$ by the formula*

$$V(x) = \begin{cases} \sum_{n=1}^{N+2} C_n e^{\beta_n x} & \text{if} \quad x \in (h_1, h_2) \\ g(x) & \text{if} \quad x \in (h_1, h_2)^c \end{cases}$$

*where $g$ is the function in (3.2). Then $V$ is the value function of the optimal stopping problem (1.1). Also, we have $V(x) = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ for all $x \in \mathbb{R}$ and hence $\tau_{(h_1,h_2)^c}$ is the optimal stopping time for the optimal stopping problem (1.1).*

**Proof :** Clearly the function $V(x)$ satisfies conditions (a)-(c) of Theorem 2.1. Direct computation shows that the function $V$ is a solution of the boundary value problem (2.7). Because $C_n$'s are nonnegative according to Lemma 3.3, thus, $h_1 < l_1 = \ln K_1 \leq \ln K_2 = l_2 < h_2$ by (3.5) and (3.6). Also functions $g_1$ and $g_2$ satisfy the conditions in Proposition 2.1. Therefore we have $V(x) = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ for all $x \in \mathbb{R}$. Note that functions $g_1$ and $g_2$ also satisfy the conditions in Proposition 2.2 and $V$ also satisfies conditions (c) and (d) of Proposition 2.2. Hence by Proposition 2.2, we obtain $\sum_{n=1}^{N+2} C_n e^{\beta_n x} \geq g(x)$ for $x \in (h_1, h_2)$. By Theorem 2.1, it remains to show that $(\mathcal{L}_X - r)V(x) \leq 0$ for $x \in [h_1, h_2]^c$. Note that, on $x > h_2 > \ln K_2$, direct calculation gives

$$\begin{aligned}
(\mathcal{L}_X - r)V(x) &= e^x(\frac{1}{2}\sigma^2 + c + \sum_{i=1}^{N^+} \frac{\lambda p_i}{\eta_i^+ - 1} + \sum_{j=1}^{N^-} \frac{\lambda q_j}{\eta_j^- - 1}) - r(e^x - K_2) \\
&+ \lambda \sum_{j=1}^{N^-} q_j e^{\eta_j^- x} \left( \sum_{n=1}^{N+2} \frac{C_n \eta_j^-}{\eta_j^- - \beta_n} e^{(\beta_n - \eta_j^-)h_2} - \frac{\eta_j^-}{\eta_j^- - 1} e^{(1-\eta_j^-)h_2} + K_2 e^{-\eta_j^- h_2} \right) \\
&= e^x(\frac{1}{2}\sigma^2 + c + \sum_{i=1}^{N^+} \frac{\lambda p_i}{\eta_i^+ - 1} + \sum_{j=1}^{N^-} \frac{\lambda q_j}{\eta_j^- - 1}) - r(e^x - K_2) \\
&+ \lambda \sum_{j=1}^{N^-} q_j e^{\eta_j^- x} \left( \sum_{n=1}^{N+2} \frac{C_n \beta_n}{\eta_j^- - \beta_n} e^{(\beta_n - \eta_j^-)h_2} - \frac{1}{\eta_j^- - 1} e^{(1-\eta_j^-)h_2} \right)
\end{aligned}$$

(The last equality holds because of (3.5).) Let $\Psi_j(x) = \sum_{n=1}^{N+2} \frac{C_n \beta_n}{\eta_j^- - \beta_n} e^{(\beta_n - \eta_j^-)x} - \frac{1}{\eta_j^- - 1} e^{(1-\eta_j^-)x}$ for $1 \leq j \leq N^-$ and $x \in \mathbb{R}$. First we show that $\Psi_j(h_2) \geq 0$. By (3.4) and (3.6), we have $\Psi_j(h_1) = \frac{2}{1-\eta_j^-} e^{(1-\eta_j^-)h_1} > 0$. Also, we observe $\Psi_j'(x) = -\sum_{n=1}^{N+2} C_n \beta_n e^{(\beta_n - \eta_j^-)x} + e^{(1-\eta_j^-)x} = -e^{-\eta_j^- x} \left( \sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x \right)$. We need the fact that $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x \neq 0$ for all $x \in (h_1, h_2)$. (Indeed, if $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n h*} - e^{h*} = 0$ for some $h^* \in (h_1, h_2)$, by Lemma 3.4, $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x \geq 0$ for all $x \in [h*, h_2]$. Note that by (3.7), we have $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n h_2} - e^{h_2} = 0$ and by Lemma 3.4, there exists $\epsilon > 0$ such that $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x < 0$ for all $x \in (h_2 - \epsilon, h_2]$ which is a contradiction.) Combining this with the fact that $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n h_1} - e^{h_1} = -2e^{h_1} < 0$, we obtain $\sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x \leq 0$ for all $x \in [h_1, h_2]$ and hence, $\Psi_j'(x) \geq 0$ on $[h_1, h_2]$. This implies that $\Psi_j(x)$ is an increasing function and hence $\Psi_j(h_2) \geq \Psi_j(h_1) > 0$. Therefore, on $x > h_2 > \ln K_2$ , we observe $\frac{d}{dx}(\mathcal{L}_X - r)V(x) = (\frac{1}{2}\sigma^2 + c + \sum_{i=1}^{N^+} \frac{\lambda p_i}{\eta_i^+ - 1} + \sum_{j=1}^{N^-} \frac{\lambda q_j}{\eta_j^- - 1} - r)e^x + \lambda \sum_{j=1}^{N^-} q_j \Psi_j(h_2) \eta_j^- e^{\eta_j^- x} \leq 0$, which implies that $(\mathcal{L}_X - r)V(x)$ is a decreasing function and its maximum value is $(\mathcal{L}_X - r)V(h_2+)$. Because $V(x)$ satisfies the smooth pasting condition at $h_2$ and $(\mathcal{L}_X - r)V(h_2-) = 0$, we get

$$\begin{aligned}
(\mathcal{L}_X - r)V(h_2+) &= (\mathcal{L}_X - r)V(h_2+) - (\mathcal{L}_X - r)V(h_2-) = \frac{1}{2}\sigma^2(V''(h_2+) - V''(h_2-)) \\
&= \frac{1}{2}\sigma^2(e^{h_2} - \sum_{n=1}^{N+2} C_n \beta_n^2 e^{\beta_n h_2}) < \frac{1}{2}\sigma^2(e^{h_2} - \sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n h_2}) = 0
\end{aligned}$$

Therefore $(\mathcal{L}_X - r)V(x) \leq (\mathcal{L}_X - r)V(h_2^+) < 0$ for all $x > h_2$. By the same procedure, we verify $(\mathcal{L}_X - r)V(x)$ is an increasing function for $x \leq h_1$ and $(\mathcal{L}_X - r)V(h_1-) \leq 0$, which implies $(\mathcal{L}_X - r)V(x) \leq 0$ for all $x \leq h_1$. The proof is complete. □

# 4 Existence of Solutions to Equations (3.3)-(3.8)

In this section we prove the existence of solutions to the system of equations (3.3)-(3.8). According to (3.25)-(3.28), we have $\widetilde{A}DC = \widetilde{K}$ where $D$ is an $(N+2)\times(N+2)$ diagonal matrix with entries $d_{ii} = \beta_i(1-\beta_i)$, $\widetilde{K} = [0,0,...,0,K_1]^T$ is an $(N+2)\times 1$ column vector and

$$\widetilde{A} = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2}h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2}h_1} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}h_2} \\ \frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1 h_1}+\frac{1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2}h_1}+\frac{1}{K_2}e^{\beta_{N+2}h_2}) \\ \frac{1}{\beta_1}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}}e^{\beta_{N+2}h_1} \end{bmatrix}.$$

Then, the coefficient vector C is equal to

$$C = \frac{K_1}{det\widetilde{A}}D^{-1}Y \tag{4.1}$$

where $Y$ is the last column of the cofactor matrix of $\widetilde{A}$. Thus, if we find out the boundaries of the continuation region $(h_1,h_2)$, then we can compute the coefficient vector $C$ by (4.1).

**Proposition 4.1** *Let $\{C_1,...,C_{N+2},h_1,h_2\}$ be a solution of the equations (3.3)-(3.8). Then $\Delta h = h_2 - h_1$ is a solution of the equation $detB(h) = 0$ where for every $h \in \mathbb{R}$, $B(h)$ is a $(N+2)\times(N+2)$ matrix defined by the formula*

$$B(h) = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}h}) \\ \frac{1}{\beta_1-1}(1+e^{(\beta_1-1)h}) & \cdots & \frac{1}{\beta_{N+2}-1}(1+e^{(\beta_{N+2}-1)h}) \end{bmatrix} \tag{4.2}$$

*Moreover, we have*

$$h_1 = \log detA_1 - \log detA_2, \tag{4.3}$$

*where*

$$A_1 = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N+2}-1} \end{bmatrix}$$

*and*

$$A_2 = \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1 K_1} & \cdots & \frac{1}{\beta_{N+2}K_1} \end{bmatrix}$$

**Proof :** See the Appendix. □

**Proposition 4.2** *Given any $h \in \mathbb{R}$, define the matrix $B(h)$ as in (4.2). There exists a positive solution $\Delta h$ to the equation $detB(h) = 0$.*

**Proof :** See the Appendix. □

**Theorem 4.1** *Let $\Delta h$ be a positive solution of the equation $detB(h) = 0$ and define $h_1$ by (4.3). Set $h_2 = h_1 + \Delta h$ and compute $\{C_1, \cdots, C_{N+2}\}$ by the formula (4.1). Then $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ is a solution of the equations (3.3)-(3.8).*

**Proof :** The system of equations (3.3)-(3.8) is equivalent to $\widetilde{A}DC = \widetilde{K}$ together with the smooth pasting conditions (3.7) and (3.8). From the proof of Proposition 4.1, we know that $\{C_1, \cdots, C_{N+2}, h_1, h_2\}$ satisfies $\widetilde{A}DC = \widetilde{K}$ and (3.8). It remains to check that (3.7) is satisfied. By (4.1), the left hand side of (3.7) is

$$
\sum_{n=1}^{N+2}\frac{K_1 y_n}{det(\widetilde{A})(1-\beta_n)}e^{\beta_n h_2} = \quad det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2} h_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2} h_1} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h_2} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h_2} \\
\frac{1}{\beta_1}(e^{\beta_1 h_1}+\frac{K_1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(e^{\beta_{N+2} h_1}+\frac{K_1}{K_2}e^{\beta_{N+2} h_2}) \\
\frac{e^{\beta_1 h_2}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} h_2}}{1-\beta_{N+2}}
\end{bmatrix}
$$

$$
\times det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2} h_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2} h_1} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h_2} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h_2} \\
\frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1 h_1}+\frac{1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2} h_1}+\frac{1}{K_2}e^{\beta_{N+2} h_2}) \\
\frac{1}{\beta_1}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}}e^{\beta_{N+2} h_1}
\end{bmatrix}^{-1}
$$

$$
= det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{e^{\beta_1 \Delta h}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} \Delta h}}{1-\beta_{N+2}}
\end{bmatrix}
$$

$$
\times det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{1}{\beta_1 K_1} & \cdots & \frac{1}{\beta_{N+2} K_1}
\end{bmatrix}^{-1}
$$

$$
= det(A_2)^{-1} det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{e^{\beta_1 \Delta h}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} \Delta h}}{1-\beta_{N+2}}
\end{bmatrix}
$$

Since $\Delta h$ satisfies $detB(h) = 0$, we have

$$-detA_1 = -det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N+2}-1} \end{bmatrix}$$

$$= det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1}e^{(\beta_1-1)\Delta h} & \cdots & \frac{1}{\beta_{N+2}-1}e^{(\beta_{N+2}-1)\Delta h} \end{bmatrix}$$

$$= e^{-\Delta h} det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1\Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}\Delta h}) \\ \frac{1}{\beta_1-1}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}-1}e^{\beta_{N+2}\Delta h} \end{bmatrix}$$

Therefore, the left hand side of (3.7) is equal to $detA_2^{-1}detA_1e^{\Delta h} = e^{h_1+\Delta h} = e^{h_2}$. The proof is complete. □

# 5 Numerical Results

In this section, we solve the system of equations (3.3)-(3.8) numerically. To solve the equations (3.3)-(3.8) numerically, we first find the length of the optimal interval $\Delta h$ by solving the equation $detB(h) = 0$ where $B(h)$ is the square matrix in (4.2). (Note that the equation above depends only on $h$ and hence we can use the simple and fast approach like the Newton method to solve it.) Second, we compute $h_1$ by (4.3) and set $h_2 = h_1 + \Delta h$. Finally, we obtain the coefficient vector C according to (4.1) and evaluate the value function $V(x)$ by the formula $V(x) = \sum_{n=1}^{N+2} C_n e^{\beta_n x}$ for $x \in (h_1, h_2)$.

**Example 3.** Consider the case that $N^+ = N^- = 1$. In addition, as in Boyarchenko [3], we take $c = -0.105$, $\sigma = 0.25$, $r = 0.06$, $\eta^+ = \frac{1}{0.4}$, $\eta^- = -\frac{1}{0.7}$, $\lambda = \frac{3}{5}$, $p = q = 0.5$ and the strike prices $K_1 = 50$ and $K_2 = 100$. Then the value function is given by $V(x) = \sum_{n=1}^{4} C_n e^{\beta_n x}$ in $(h_1^*, h_2^*)$ where

$$\begin{aligned} (h_1^*, h_2^*) &= (2.1992, 6.1953) \\ \{\beta_1, \beta_2, \beta_3, \beta_4\} &= \{-3.4812, -0.2322, 1.1995, 6.953\} \\ \{C_1, C_2, C_3, C_4\} &= \{2519.533, 61.2124, 0.2183, 1.4624 \times 10^{-18}\}. \end{aligned}$$

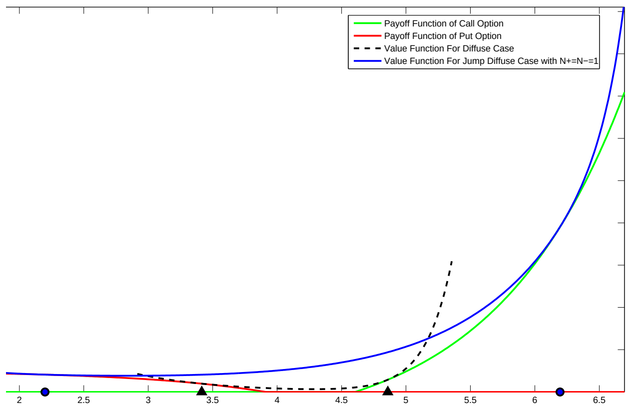


Figure 2: The solid line is the value function $V(x)$ for the jump-diffusion model with $N^+ = N^- = 1$ and the dash line is the one for the diffusion model, that is, $N^+ = N^- = 0$. The optimal boundaries are marked by circles for jump-diffusion model, and by triangles for diffusion model.

Besides, if we take $N^+ = N^- = 0$ which is the diffusion case in Example 1, then we observe $V(x) = \sum_{n=1}^{2} C_n e^{\beta_n x}$ in $(h_1^*, h_2^*)$ where

$$\begin{aligned} (h_1^*, h_2^*) &= (3.4151, 4.859) \\ \{\beta_1, \beta_2\} &= \{-1.5607, 4.9207\} \\ \{C_1, C_2\} &= \{4037.8534, 1.1088 \times 10^{-9}\}. \end{aligned}$$

It is interesting to note that in the jump-diffusion model, the optimal interval $(h_1^*, h_2^*)$ is much wider than that for the diffusion case. This indeed makes sense because there are more opportunities to earn large gains by the jump occurring and hence it can be expected that the investors will not exercise the options in the jump-diffusion environment earlier than in the diffusion one. Figure 3 shows the graph of the determinant of $B(h)$ as a function of $h$. It shows that the zero of the determinant (this is $\Delta h$) is unique. Besides, the graph descends sharply near the zero of the determinant. This implies that we can get the numerical result for $\Delta h$ fast and correctly. □

**Example 4.** Consider the jump-diffusion model with $N^- = N^+ = 2$ and let $c = -0.105$, $\sigma = 0.25$, $r = 0.06$, $\eta_1^+ = \frac{1}{0.5}$, $\eta_2^+ = \frac{1}{0.25}$, $\eta_1^- = -\frac{1}{2.4}$,$\eta_2^- = -7.5$, $\lambda = \frac{3}{5}$, $p_1 = p_2 = q_1 = q_2 = 0.25$ and the strike prices $K_1 = 50$ and $K_2 = 100$. In this model, the expected value $E[e^{X_1}]$ is the same as the one with $N^- = N^+ = 1$ in Example 3. The value function is $V(x) = \sum_{n=1}^{6} C_n e^{\beta_n x}$ in $(h_1^*, h_2^*)$ where

$$\begin{aligned} (h_1^*, h_2^*) &= (2.1153, 6.3801) \\ \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\} &= \{-7.997, -1.9409, -0.1155, 1.1642, 3.2421, 7.0931\} \\ \{C_1, C_2, C_3, C_4, C_5, C_6\} &= \{735200.1029, 240.6048, 44.1297, 0.2679, 8.8413 \times 10^{-9}, \\ &\quad 2.4671 \times 10^{-19}\}. \end{aligned}$$

As noted before, models in Example 3 and Example 4 have the same expected value $E[e^{X_1}]$. However the optimal interval in Example 4 ($N^- = N^+ = 2$) is wider than that for the case $N^- = N^+ = 1$. □

## 6 Concluding Remarks

American option contracts are more complicated to analyze than their European counterparts, because an American option can be exercised at any time prior to its expiration. Mathematically this means that we have to solve the optimal stopping problem of the form in (1.1). Instead of the corresponding PDEs for the European counterparts, problem of this kind always leads to so

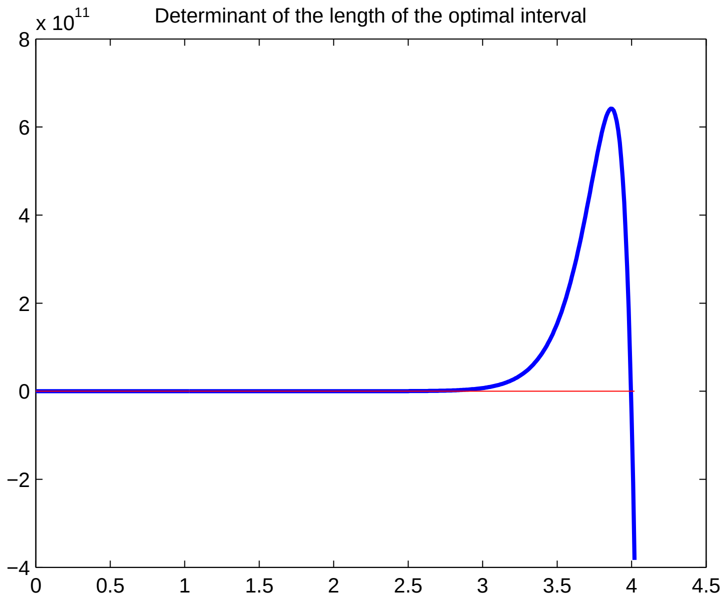


Figure 3: The figure is the graph of the determinant $B(h)$ for finding the length $\Delta h$ of the optimal interval. It shows that there is only one zero for the determinant.

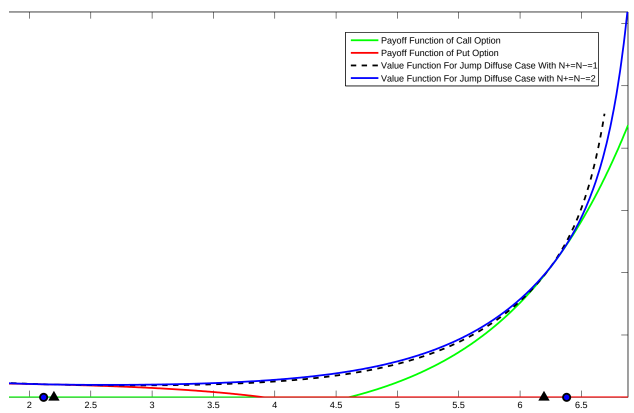


Figure 4: The solid line is the value function $V(x)$ for the jump-diffusion model with $N^+ = N^- = 2$ and the dash line is the one for the model with $N^+ = N^- = 1$. The optimal boundaries for the case $N^+ = N^- = 2$ are marked by circles and by triangles for the case $N^- = N^+ = 1$.

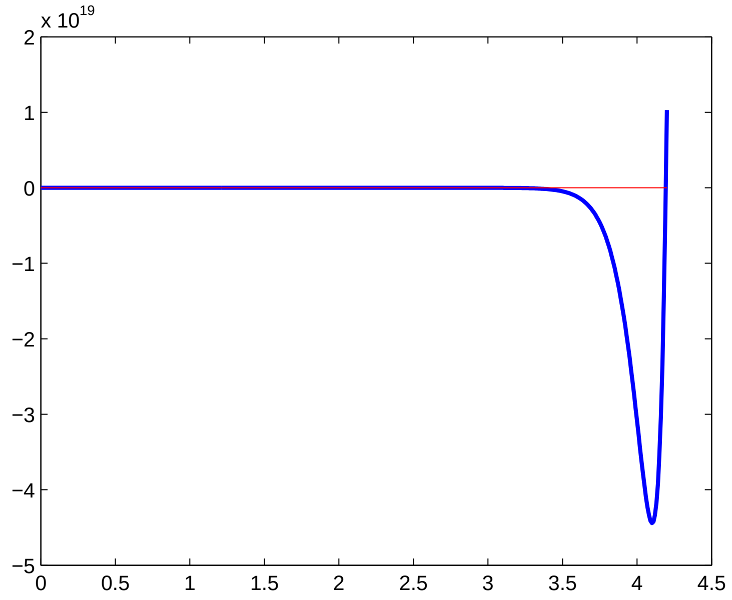


Figure 5: The figure is the graph of the determinant for finding the length of the optimal interval for the case $N^- = N^+ = 2$. The figure has similar properties as for the case $N^- = N^+ = 1$. In particular, there is only one zero for the determinant.

called free boundary value problems, that is not easy to solve. Usually we have no explicit pricing formulas for the value functions and the optimal exercise boundaries are not known. We refer to the monograph of Peskir and Shiryaev [15] for more details and related topics about optimal stopping and free boundary problem.

The American call and put options are the simplest American contracts. The pricing problem for these options has been widely studied and generalized since Mckean [10] and Merton [11]. For recent works on Lévy-model setting, we refer to Mordecki and Salminen [12], Boyarchenko and Levendorskii [4] and Asmussen et al. [1] and the references therein.

In this paper we consider the perpetual American strangle and straddle options, which is a combination of a put and a call written on the same security. As in Asmussen et al. (2004) and many others, we consider the pricing problems of these options in the jump diffusion models. By the free boundary problem approach, we solve the corresponding optimal stopping problems and hence find the optimal exercise boundaries and the rational prices of the perpetual American strangle and straddle options. More precisely, following the approach in Chen et al [5], we derive an equivalent system of equations for the free boundary problem with smooth pasting condition. By solving the system of equations, we find an algorithm for computing the rational prices and the optimal exercise boundaries for these options. (Boyarchenko [3] studied the same pricing problems by a different approach and assuming the smooth pasting principle for the value functions. In fact Boyarchenko posted the verification of the smooth pasting principle for the value functions as an open problem in [3] and we resolve this open problem in Theorem 3.1.) The present method together with the general results in Section 2 could possibly give an alternative approach to compute prices for other exotic options in jump diffusion models.

# 7 Appendix

**Proof of Proposition 2.1.** We follow similar argument as that in Chen et al. [5]. Fix $x \in (h_1, h_2)$. Pick a sequence of functions $\{\widetilde{V}_n\} \subset \mathcal{C}_0^2(\mathbb{R})$ such that $\widetilde{V}_n \equiv \widetilde{V}$ on $[h_1, h_2]$ and $\widetilde{V}_n \to \widetilde{V}$ on $\mathbb{R}$. Since $g_1$ is bounded, we can choose $\{\widetilde{V}_n\}$ such that $\{\widetilde{V}_n\}$ are uniformly bounded on $(-\infty, c]$ for any $c \in \mathbb{R}$, and $\widetilde{V}_n(x) \leq 2g_2(x)$ for all $n$ and all $x \geq M$. Here $M > h_2$ is a strictly positive constant(independent of $n$). By Dynkin's formula, we have

$$\mathbb{E}_x\left[e^{-r(t\wedge\tau_I)}\widetilde{V}_n(X_{\tau_I\wedge t})\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_I} e^{-ru}(\mathcal{L}_X - r)\widetilde{V}_n(X_u)du\right] + \widetilde{V}(x). \tag{7.1}$$

For every $u < \tau_I \wedge t$, we have $X_u \in (h_1, h_2)$ and hence $\widetilde{V}_n(X_u) = \widetilde{V}(X_u)$. This gives

$$(\mathcal{L}_X - r)\widetilde{V}(X_u) - (\mathcal{L}_X - r)\widetilde{V}_n(X_u) = \int \left[\widetilde{V}(X_u + y) - \widetilde{V}_n(X_u + y)\right] f(y)dy \tag{7.2}$$

and hence

$$\left|(\mathcal{L}_X - r)[\widetilde{V}(X_u) - \widetilde{V}_n(X_u)]\right| \leq \sup_{z\leq M+|h_1|+|h_2|}\left[|\widetilde{V}(z)| + |\widetilde{V}_n(z)|\right] + \sup_{h_1\leq x\leq h_2}\int_{M+|h_1|}^{\infty} 3g_2(x+y)f(y)dy < \infty. \tag{7.3}$$

By Dominated Convergence Theorem and (7.2), for all $u < t\wedge\tau_I$, $(\mathcal{L}_X - r)\widetilde{V}_n(X_u) \to (\mathcal{L}_X - r)\widetilde{V}(X_u)$ as $n \to \infty$. By (7.3) and Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau_I} e^{-ru}(\mathcal{L}_X - r)\widetilde{V}_n(X_u)du\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_I} e^{-ru}(\mathcal{L}_X - r)\widetilde{V}(X_u)du\right].$$

Note that $|\widetilde{V}_n(x)| \leq \sup_n \sup_{x\leq M}|\widetilde{V}_n(x)| + 2g(x)$ and $\mathbb{E}_x\left[\sup_{t\geq 0} e^{-rt}g(X_t)\right] < \infty$. Letting $n \to \infty$ for both sides of (7.1) together with the dominated convergence theorem gives

$$\mathbb{E}_x\left[e^{-r(t\wedge\tau_I)}\widetilde{V}(X_{\tau_I\wedge t})\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau_I} e^{-ru}(\mathcal{L}_X - r)\widetilde{V}(X_u)du\right] + \widetilde{V}(x) = \widetilde{V}(x). \tag{7.4}$$

Note that the last equality follows from the assumption that $(\mathcal{L}_X - r)\widetilde{V} = 0$ in $(h_1, h_2)$. Since

$$\mathbb{E}_x\left[\sup_{t\geq 0} e^{-rt}\widetilde{V}(X_t)\right] \leq \sup_{y\leq h_2}\widetilde{V}(y) + \mathbb{E}_x\left[\sup_{t\geq 0} e^{-rt}g(X_t)\right] < \infty,$$

our result follows by letting $t \to \infty$ in both sides of the equality in (7.4) and the Dominated Convergence Theorem. This completes the proof. □

**Proof of Lemma 3.1.** First consider the case that $N^+ = 0$. Then $\beta_{N^-+2}$ is the unique solution to the equation $\phi(x) = 0$ in $(0, \infty)$. Observe that $\lim_{x\to\infty}\phi(x)\lim_{x\to 1}\phi(x) = -\infty$. Our result follows by the intermediate value theorem. Next assume that $N^+ \geq 1$. Then $\beta_{N^-+2}$ is the unique solution to the equation

$$\phi(x) = \prod_{i=1}^{N^+}(\eta_i^+ - x)\prod_{j=1}^{N^-}(\eta_j^- - x)\left[\frac{1}{2}\sigma^2x^2 + cx - (\lambda + r) + \lambda\left(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - x} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - x}\right)\right] = 0$$

in $(0, \eta_1^+)$. Also we have $\phi(1) = \prod_{i=1}^{N^+}(\eta_i^+ - 1)\prod_{j=1}^{N^-}(\eta_j^- - 1)\left[\frac{1}{2}\sigma^2 + c - (\lambda + r) + \lambda(\sum_{i=1}^{N^+}\frac{p_i\eta_i^+}{\eta_i^+ - 1} + \sum_{j=1}^{N^-}\frac{q_j\eta_j^-}{\eta_j^- - 1})\right]$ and $\phi(\eta_1^+) = \lambda p_1\eta_1^+\prod_{i=2}^{N^+}(\eta_i^+ - \eta_1^+)\prod_{j=1}^{N^-}(\eta_j^- - \eta_1^+)$. By (3.9) and (3.10), we obtain $\phi(1)\phi(\eta_1^+) < 0$ which implies $\beta_{N^-+2} > 1$. □

**Proof of Lemma 3.2.** Set $\Delta h = h_2 - h_1$ and put $\widehat{C}_n = e^{\beta_n h_1}(1 - \beta_n)\beta_n C_n$ for $1 \leq n \leq N + 2$. Then, by(3.24)-(3.27), we have $A\widehat{C} = K$ where

$$A = \begin{bmatrix} \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^-}^-} \\ \frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N+2}} \\ \frac{1}{\beta_1}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2}}e^{\beta_{N+2}\Delta h} \\ \frac{1}{\beta_1 - \eta_1^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+}e^{\beta_{N+2}\Delta h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^+}^+}e^{\beta_1\Delta h} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^+}^+}e^{\beta_{N+2}\Delta h} \end{bmatrix}, \widehat{C} = \begin{bmatrix} \widehat{C}_1 \\ \widehat{C}_2 \\ \vdots \\ \widehat{C}_{N+2} \end{bmatrix} \text{ and } K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K_1 \\ -K_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let $F_1(x) = \sum_{i=1}^{N+2} \frac{\widehat{C}_i}{\beta_i - x}$ and $F_2(x) = \sum_{i=1}^{N+2} \frac{e^{\beta_i \Delta h}\widehat{C}_i}{\beta_i - x}$. Clearly, $F_1(x) = \frac{S_1(x)}{\prod_{i=1}^{N+2}(\beta_i - x)}$ and $F_2(x) = \frac{S_2(x)}{\prod_{i=1}^{N+2}(\beta_i - x)}$, where

$$S_1(x) = \sum_{n=1}^{N+2} \widehat{C}_n \prod_{i=1, i\neq n}^{N+2} (\beta_i - x) \quad \text{and} \quad S_2(x) = \sum_{n=1}^{N+2} e^{\beta_n \Delta h}\widehat{C}_n \prod_{i=1, i\neq n}^{N+2} (\beta_i - x). \tag{7.5}$$

Then $S_1(x)$ and $S_2(x)$ are polynomials with degree at most $N+1$. Also, by the fact $A\widehat{C} = K$, we have $S_1(0) = K_1 \prod_{i=1}^{N+2} \beta_i$, $S_2(0) = -K_2 \prod_{i=1}^{N+2} \beta_i$, $S_1(\eta_k^-) = 0$ for $1 \le k \le N^-$ and $S_2(\eta_k^+) = 0$ for $1 \le k \le N^+$. By (7.5), we have

$$\widehat{C}_n = \frac{S_1(\beta_n)}{\prod_{i=1, i\neq n}^{N+2}(\beta_i - \beta_n)} = \frac{e^{-\beta_n \Delta h} S_2(\beta_n)}{\prod_{i=1, i\neq n}^{N+2}(\beta_i - \beta_n)} \tag{7.6}$$

for $1 \le n \le N+2$. From this, we have $S_2(\beta_n) = S_1(\beta_n) = 0$ if and only if $S_2(\beta_n) - S_1(\beta_n) = 0$. In addition, we have $\widehat{C}_n = 0$ if and only if $S_2(\beta_n) - S_1(\beta_n) = 0$. Also if $S_1(\beta_k)$ and $S_2(\beta_k)$ are nonzero for some $1 \le k \le N+2$, $\frac{S_2(\beta_k)}{S_1(\beta_k)} = e^{\beta_k \Delta h}$. It remains to show that $|\Theta| \le 1$ where $\Theta = \{\beta_n | S_1(\beta_n) - S_2(\beta_n) = 0, \text{ for } 1 \le n \le N+2\}$ and $|\Theta|$ is the cardinality of $\Theta$. To do this, we need the following facts :

(1) If $S_2(x) \neq 0$ on $(\eta_k^-, \beta_{k+1}]$ for some $k$, $1 \le k \le N^-$, then $S_2(x) - S_1(x) = 0$ has a solution in $(\eta_k^-, \beta_{k+1})$.

(2) If $S_2(x) \neq 0$ on $[\beta_k, \eta_k^-)$ for some $k$, $1 \le k \le N^-$, then $S_2(x) - S_1(x) = 0$ has a solution in $(\beta_k, \eta_k^-)$ .

(3) If $S_1(x) \neq 0$ on $(\eta_k^+, \beta_{N^-+2+k}]$ for some $k$, $1 \le k \le N^+$, then $S_2(x) - S_1(x) = 0$ has a solution in $(\eta_k^+, \beta_{N^-+2+k})$.

(4) If $S_1(x) \neq 0$ on $[\beta_{N^-+1+k}, \eta_k^+)$ for some $k$, $1 \le k \le N^+$, then $S_2(x) - S_1(x) = 0$ has a solution in $(\beta_{N^-+1+k}, \eta_k^+)$.

(5) If $S_2(x) \neq 0$ on $[\beta_{N^-+1}, 0)$, then $S_1(x)$ has a solution in $(\beta_{N^-+1}, 0)$.

(6) If $S_1(x) \neq 0$ on $(0, \beta_{N^-+2}]$, then $S_2(x)$ has a solution in $(0, \beta_{N^-+2})$.

To prove (1), we assume that $S_2(x) \neq 0$ for all $x \in (\eta_k^-, \beta_{k+1}]$. Let $x^* = \sup\{x \in [\eta_k^-, \beta_{k+1}] | S_1(x) = 0\}$. Note that $x^*$ exists because $S_1(\eta_k^-) = 0$ and $x^* < \beta_{k+1}$. Because $\frac{S_2(x)}{S_1(x)}$ is continuous on $(x^*, \beta_{k+1}]$, $0 < \frac{S_2(\beta_{k+1})}{S_1(\beta_{k+1})} = e^{\beta_{k+1}\Delta h} < 1$ and $\lim_{x \to x^{*+}} \frac{S_2(x)}{S_1(x)} = \infty$, by the intermediate value theorem, there exists $x_0^* \in (x^*, \beta_{k+1})$ such that $\frac{S_2(x_0^*)}{S_1(x_0^*)} = 1$. This completes the proof of the fact (1) above. Facts (2)-(4) are verified by similar arguments.

Next , we verify the fact (5) and assume that $S_2(x) \neq 0$ for all $x \in [\beta_{N^-+1}, 0)$. Then

$$sgn(S_2(\beta_{N^-+1})S_1(\beta_{N^-+1})) = sgn\left(e^{\beta_{N^-+1}\Delta h}\widehat{C}_n^2 \prod_{i=1, i\neq N^-+1}^{N+2} (\beta_i - \beta_{N^-+1})^2\right) > 0,$$

and $sgn(S_2(0)S_1(0)) = sgn(-K_1 K_2 \prod_{i=1}^{N+2} \beta_i^2) < 0$, which imply that $S_1(x)$ has a solution in $(\beta_{N^-+1}, 0)$. The proof of the fact (6) is similar.

Let $S(x) = S_2(x) - S_1(x)$. Then $S(x)$ is a polynomial with degree at most $N+1$ and $S(\beta_k) = 0$ whenever $\beta_k \in \Theta$. Let

$$\begin{aligned} \Pi = \quad & \{[\beta_{N^-+1}, 0) | \beta_{N^-+1} \notin \Theta\} \cup \{(0, \beta_{N^-+2}] | \beta_{N^-+2} \notin \Theta\} \\ & \cup \{[\beta_k, \eta_k^-) | \beta_k \notin \Theta, 1 \le k \le N^-\} \cup \{(\eta_k^-, \beta_{k+1}] | \beta_{k+1} \notin \Theta, 1 \le k \le N^-\} \\ & \cup \{[\beta_{N^-+1+k}, \eta_k^+) | \beta_{N^-+1+k} \notin \Theta, 1 \le k \le N^+\} \cup \{(\eta_k^+, \beta_{N^-+2+k}] | \beta_{N^-+2+k} \notin \Theta, 1 \le k \le N^+\}. \end{aligned}$$

Note that $\Pi$ is a collection of intervals and $|\Pi| \equiv$ the number of intervals in $\Pi \ge 2(N+1) - 2|\Theta|$. Let $\widetilde{\Pi} = \{I \in \Pi | S(x) = 0 \text{ has no solution in } I\}$. Since $|\{x | S(x) = 0, x \notin \Theta\}| \le N+1-|\Theta|$, $|\widetilde{\Pi}| \ge 2(N+1) - 2|\Theta| - ((N+1) - |\Theta|) = N+1-|\Theta|$. For any $I \in \widetilde{\Pi}$, by facts (1)-(4), we obtain

(a) if $\sup_{x\in I} x \le \beta_{N^-+1}$, then the equation $S_2(x) = 0$ has solutions in $I$.

(b) if $\inf_{x\in I} x \ge \beta_{N^-+2}$, then the equation $S_1(x) = 0$ has solutions in $I$.

Also, by fact (5),$S_1(x)S_2(x) = 0$ for some $x \in [\beta_{N^-+1}, 0)$. Similarly, by fact (6), $S_1(x)S_2(x) = 0$ for some $x \in (0, \beta_{N^-+2}]$. From these observation, combining with the fact that for $I_1, I_2 \in \widetilde{\Pi}$, $I_1 \cap I_2 = \phi$ or $I_1 \cap I_2 \subseteq \Theta^c$, we have

$$|\{x|S_2(x) = 0, x < \beta_{N^-+2}, x \notin \Theta\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}, x \notin \Theta\}| \ge |\widetilde{\Pi}| \ge N + 1 - |\Theta|. \quad (7.7)$$

Recall that $S_1(\eta_k^-) = 0$ for $1 \le k \le N^-$ and $S_2(\eta_k^+) = 0$ for $1 \le k \le N^+$. Therefore,

$$\begin{aligned}
&2(N+1) \ge |\{x|S_1(x) = 0\}| + |\{x|S_2(x) = 0\}| \\
&= \left|\{x|S_1(x) = 0, x > \beta^-_{N^-+1}, x \notin \Theta\}\right| + \left|\{x|S_1(x) = 0, x \le \beta^-_{N^-+1}, x \notin \Theta\}\right| \\
&+ \left|\{x|S_2(x) = 0, x < \beta^+_{N^-+2}, x \notin \Theta\}\right| + \left|\{x|S_2(x) = 0, x \ge \beta^+_{N^-+2}, x \notin \Theta\}\right| + 2|\{x|x \in \Theta\}| \\
&\ge N + 1 - |\Theta| + N^- + N^+ + 2|\Theta| = 2N + 1 + |\Theta|. \quad (7.8)
\end{aligned}$$

This implies that $|\Theta| \le 1$. The proof is complete. □

**Proof of Lemma 3.3.** We define $S_1$, $S_2$, $\Theta$, $\widetilde{\Pi}$, $\Pi$, and $\widehat{C}_n$'s as in the proof of Lemma 3.2. Since $\widehat{C}_n = e^{-\beta_n h_1}(1-\beta_n)\beta_n C_n$ and, by Lemma 3.1, we observe $C_n \ge 0$ if and only if $\widehat{C}_n \le 0$. Besides, by Proposition (2.1), we obtain $\sum_{n=1}^{N+2} C_n e^{\beta_n x} = \mathbb{E}_x[e^{-r\tau_{(h_1,h_2)^c}} g(X_{\tau_{(h_1,h_2)^c}})]$ which is nonnegative for all $x \in (h_1, h_2)$. To prove $C_n \ge 0$ for all $n$, it suffices to show that the $\widehat{C}_n$'s have the same sign. By Lemma (3.2), $|\Theta| = 0$ or 1. First, we consider the case that $|\Theta| = 1$, that is, $S_1(\beta_{k_0}) = S_2(\beta_{k_0}) = 0$ for some $1 \le k_0 \le N + 2$. Then $|\Pi| \ge 2N$ and by (7.7),

$$|\{x|S_2(x) = 0, x < \beta_{N^-+2}, x \neq \beta_{k_0}\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}, x \neq \beta_{k_0}\}| \ge |\widetilde{\Pi}| \ge N + 1 - 1 = N.$$

By (7.8), we obtain $|\{x|S_2(x) = 0\}| + |\{x|S_1(x) = 0\}| = 2N + 2$. Hence $S_1(x)$ and $S_2(x)$ are polynomials with degree $N + 1$ and all roots of $S_1(x)$ and of $S_2(x)$ are simple. In addition

$$\begin{aligned}
& 2(N+1) \ge |\{x|S_1(x) = 0\}| + |\{x|S_2(x) = 0\}| \\
\ge\ & |\{x|S_2(x) = 0, x < \beta_{N^-+2}, x \neq \beta_{k_0}\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}, x \neq \beta_{k_0}\}| \\
& + |\{x|S_2(x) = 0, x \ge \beta_{N^-+2}, x \neq \beta_{k_0}\}| + |\{x|S_1(x) = 0, x \le \beta_{N^-+1}, x \neq \beta_{k_0}\}| + 2 \\
\ge\ & N + |\{x|S_2(x) = 0, x \ge \beta_{N^-+2}, x \neq \beta_{k_0}\}| + |\{x|S_1(x) = 0, x \le \beta_{N^-+1}, x \neq \beta_{k_0}\}| + 2
\end{aligned}$$

and hence, $N \ge |\{x|S_2(x) = 0, x \ge \beta_{N^-+2}, x \neq \beta_{k_0}\}| + |\{x|S_1(x) = 0, x \le \beta_{N^-+1}, x \neq \beta_{k_0}\}|$. Since $S_2(\eta_k^+) = 0$ for $1 \le k \le N^+$ and $S_1(\eta_k^-) = 0$ for $1 \le k \le N^-$, we obtain $\{x|S_1(x) = 0, x \le \beta^-_{N^-+1}, x \neq \beta_{k_0}\} = \{\eta_k^-|1 \le k \le N^-\}$ and $\{x|S_2(x) = 0, x \ge \beta_{N^-+2}, x \neq \beta_{k_0}\} = \{\eta_k^+|1 \le k \le N^+\}$. Now we consider the case that $k_0 = 1$, that is $S_1(\beta_1) = S_2(\beta_1) = 0$. Because $\eta_i^-$ is the unique root for $S_1(x)$ in $[\beta_i, \beta_{i+1}]$, $2 \le i \le N^-$, we obtain $S_1(\beta_i)S_1(\beta_{i+1}) < 0$. By similar arguments, we also have $S_2(\beta_j)S_2(\beta_{j+1}) < 0$ for $N^- + 2 \le j \le N + 1$. By (7.6), we have

$$\begin{aligned}
\widehat{C}_{n-1}\widehat{C}_n &= \frac{e^{-\beta_{n-1}\Delta h} S_2(\beta_{n-1})}{\prod_{i=1,i\neq n-1}^{N+2}(\beta_i - \beta_{n-1})} \frac{e^{-\beta_n \Delta h} S_2(\beta_n)}{\prod_{i=1,i\neq n}^{N+2}(\beta_i - \beta_n)} \\
&= \frac{e^{-(\beta_{n-1}+\beta_n)\Delta h} S_2(\beta_{n-1})S_2(\beta_n)(\beta_n - \beta_{n-1})^{-1}(\beta_{n-1} - \beta_n)^{-1}}{\prod_{i=1}^{n-2}(\beta_i - \beta_{n-1})(\beta_i - \beta_n)\prod_{j=n+1}^{N+2}(\beta_j - \beta_{n-1})(\beta_j - \beta_n)} \\
&= \frac{S_1(\beta_{n-1})S_1(\beta_n)(\beta_n - \beta_{n-1})^{-1}(\beta_{n-1} - \beta_n)^{-1}}{\prod_{i=1}^{n-2}(\beta_i - \beta_{n-1})(\beta_i - \beta_n)\prod_{j=n+1}^{N+2}(\beta_j - \beta_{n-1})(\beta_j - \beta_n)},
\end{aligned}$$

Therefore, the elements in $C^- \equiv \{\widehat{C}_n|2 \le n \le N^- + 1\}$ have the same sign and this is also true for elements in $C^+ \equiv \{\widehat{C}_n|N^- + 2 \le n \le N + 2\}$. Because $A\widehat{C} = K$, if the elements in $C^-$ are positive and the ones in $C^+$ are negative, then we get the contradiction that $K_1 = \sum_{n=1}^{N+2} \widehat{C}_n \frac{1}{\beta_n} < 0$; if the elements in $C^-$ are negative and the ones in $C^+$ are positive, then we get another contradiction, i.e.,

$-K_2 = \sum_{n=1}^{N+2} \widehat{C}_n \frac{e^{\beta_n \Delta h}}{\beta_n} > 0$. Therefore, $\widehat{C}_n$'s must have the same sign. For the case $k_0 = N^- + 1$, the proof is the same. For the case $1 < k_0 < N^- + 1$, by a similar argument as above, we obtain the elements in $C_1^- = \{\widehat{C}_n | 1 \leq n \leq k_0 - 1\}$, $C_2^- = \{\widehat{C}_n | k_0 + 1 \leq n \leq N^- + 1\}$, and $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively. There are eight situations for the signs of $C_1^-$, $C_2^-$, and $C^+$ : (1) $C_1^- < 0$, $C_2^- < 0$, and $C^+ < 0$, (2) $C_1^- > 0$, $C_2^- > 0$, and $C^+ > 0$, (3)$C_1^- < 0$, $C_2^- < 0$, and $C^+ > 0$, (4)$C_1^- > 0$, $C_2^- > 0$, and $C^+ < 0$, (5) $C_1^- < 0$, $C_2^- > 0$, and $C^+ > 0$, (6) $C_1^- > 0$, $C_2^- < 0$, and $C^+ < 0$, (7) $C_1^- < 0$, $C_2^- > 0$, and $C^+ < 0$, (8) $C_1^- > 0$, $C_2^- < 0$, and $C^+ > 0$. (We write $C_i^{\pm} > (<) 0$ if all elements in $C_i^{\pm}$ are greater(smaller) than zero.) We show that cases (3)-(8) are impossible. The arguments for disproving cases (3) and (4) are the same as for the case $k_0 = 1$. Note that

$$\begin{aligned}
&\beta_1 < \eta_1^- < \beta_2 < \eta_2^- < \cdots < \beta_{k_0} < \eta_{k_0}^- < \beta_{k_0} < \cdots < \beta_{N^-} < \eta_{N^-}^- < \beta_{N^-+1} < 0 < 1 \\
&< \beta_{N^-+2} < \eta_1^+ < \cdots < \beta_{N+1} < \eta_{N^+}^+ < \beta_{N+2}.
\end{aligned}$$

Because $A\widehat{C} = K$, Comparing with the $(k_0-1)$-th entries in $A\widehat{C}$ and $K$, we obtain $\sum_{n=1}^{N+2} \widehat{C}_n \frac{1}{\beta_n - \eta_{k_0-1}^-} = 0$. Therefore, it is impossible for cases (5) and (6). Note that the entries of $A$ satisfy the following:

(a) $A_{i,j} < 0$ for $\{(i,j) | 1 \leq j \leq i \leq N^- + 1\} \cup \{(i,j) | N^- + 2 \leq i \leq N + 2, 1 \leq j < i\}$ and $A_{i,j} > 0$, otherwise.

(b) If $A_{i,j}$ and $A_{i+1,j}$ are negative, then $A_{i,j} < A_{i+1,j}$.

(c) If $A_{i,j}$ and $A_{i+1,j}$ are positive, then $A_{i,j} < A_{i+1,j}$.

For the case (7), we get the contradiction $K_1 = (A_{N^-+1} - A_{k_0-1})\widehat{C} < 0$ and for the case (8), we get the contradiction $-K_2 = (A_{N^-+2} - A_{k_0-1})\widehat{C} > 0$ where $A_i$ is the ith row of A. Therefore, we complete the proof for the case that $|\Theta| = 1$ and $1 < k_0 \leq N^- + 1$. The proof for the case that $|\Theta| = 1$ and $N^- + 2 \leq k_0 \leq N + 2$ is similar.

Consider the case that $|\Theta| = 0$ which implies that $\widehat{C}_n$'s are nonzero. Then we have $|\Pi| = 2N + 2$ and by (7.7), $|\{x | S_2(x) = 0, x < \beta_{N^-+2}\}| + |\{x | S_1(x) = 0, x > \beta_{N^-+1}\}| \geq |\widetilde{\Pi}| \geq N + 1$. Therefore

$$\begin{aligned}
& 2(N+1) \geq |\{x|S_1(x) = 0\}| + |\{x|S_2(x) = 0\}| \\
\geq \quad & |\{x|S_2(x) = 0, x < \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x > \beta_{N^-+1}\}| \\
& + |\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}| + 2|\Theta| \\
\geq \quad & N + 1 + |\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}|
\end{aligned}$$

which implies $N + 1 \geq |\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}|$. Because $|\{x|S_2(x) = 0, x \geq \beta_{N^-+2}\}| + |\{x|S_1(x) = 0, x \leq \beta_{N^-+1}\}| \geq N$, we have $|\{x|x > \beta_{N^-+2}, S_2(x) = 0\}| = N^+$ or $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^-$. First, we consider the case $|\{x|x > \beta_{N^-+2}, S_2(x) = 0\}| = N^+$, or equivalently, $\{x|x \geq \beta_{N^-+2}, S_2(x) = 0\} = \{\eta_1^+ \cdots \eta_{N^+}^+\}$. If $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^-$, then we have $\{x|x \leq \beta_{N^-+1}, S_1(x) = 0\} = \{\eta_1^- \cdots \eta_{N^-}^-\}$ . Similar arguments as for the case $|\Theta| = 1$ imply that the elements in $C^- = \{\widehat{C}_n | 1 \leq n \leq N^- + 1\}$ and in $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively, and hence, the sign of $\widehat{C}_n$'s are the same. If $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^- + 1$, then either $S_1(x)$ has a root in $(-\infty, \beta_1)$ or $S_1(x)$ has two roots in $(\beta_{k_0}, \beta_{k_0+1})$ for some $1 \leq k_0 \leq N^-$. For the case $(-\infty, \beta_1)$, we can get as above that the elements in $C^- = \{\widehat{C}_n | 1 \leq n \leq N^- + 1\}$ and in $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively. If $S_1(x)$ has two roots in $(\beta_{k_0}, \beta_{k_0+1})$ for some $1 \leq k_0 \leq N^-$, we also observe that the elements in $C_1^- = \{\widehat{C}_n | 1 \leq n \leq k_0 - 1\}$, $C_2^- = \{\widehat{C}_n | k_0 \leq n \leq N^- + 1\}$, and $C^+ = \{\widehat{C}_n | N^- + 2 \leq n \leq N + 2\}$ have the same sign, respectively. By the same argument as for the case $|\Theta| = 1$, we know that the coefficients have the same sign. The proof for the case $|\{x|x < \beta_{N^-+1}, S_1(x) = 0\}| = N^-$ is similar and hence, we omit it. □

**Proof of Lemma 3.4.** Let $F(x) = \sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x} - e^x$. Then $F'(x) = \sum_{n=1}^{N+2} C_n \beta_n^2 e^{\beta_n x} - e^x$. Because $\beta_1 < \beta_2 < \cdots < \beta_{N^-+1} < 0 < 1 < \beta_{N^-+2} < \beta_{N^-+3} < \cdots < \beta_{N+2}$, and by Lemma 3.2 and Lemma 3.3,

$$F'(x_0) = \sum_{n=1}^{N+2} C_n \beta_n^2 e^{\beta_n x_0} - e^{x_0} > \sum_{n=1}^{N+2} C_n \beta_n e^{\beta_n x_0} - e^{x_0} = 0, \tag{7.9}$$

which implies that $F(x)$ is strictly increasing in some neighborhood $U_{x_0}$ of $x_0$ and hence, we complete the proof of the first part of the lemma. Assume that there exists $x' > x_0$ such that $F(x') < 0$. Let $\widehat{x} = \sup\{x|x_0 \le x < x', F(x) = 0\}$. Then $\widehat{x} < x', F(\widehat{x}) = 0$ and as shown for (7.9), we have $F'(\widehat{x}) > 0$. Therefore, there exists a neighborhood $U_{\widehat{x}}$ of $\widehat{x}$ such that for all $x \in U_{\widehat{x}}$ with $x > \widehat{x}$, $F(x) > F(\widehat{x}) = 0$. This is a contradiction because $F(x) < 0$ for all $x \in (\widehat{x}, x')$ and hence, we complete the proof of the lemma. □

**Proof of Proposition 4.1.** Substitute (4.1) into (3.29), we have

$$\sum_{n=1}^{N+2} \frac{K_1 y_n}{(1-\beta_n)detA}(e^{(\beta_n-1)h_1} + e^{(\beta_n-1)h_2}) = 0 \tag{7.10}$$

where $y_n$ is the nth entry of the column vector $Y$. (7.10) is equivalent to

$$det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2}h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2}h_1} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}h_2} \\ \frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1 h_1} + \frac{1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2}h_1} + \frac{1}{K_2}e^{\beta_{N+2}h_2}) \\ \frac{1}{1-\beta_1}(e^{(\beta_1-1)h_1} + e^{(\beta_1-1)h_2}) & \cdots & \frac{1}{1-\beta_{N+2}}(e^{(\beta_{N+2}-1)h_1} + e^{(\beta_{N+2}-1)h_2}) \end{bmatrix} = 0.$$

Multiply $e^{-\beta_i h_1}$ to the i-th column for each $i$ and then $-e^{h_1}$ to the last row, we observe that $\Delta h = h_2 - h_1$ is a solution of the equation $detB(h) = 0$. Substitute (4.1) into (3.8), we have $\frac{K_1}{det(A)}[\beta_1 e^{\beta_1 h_1}, ..., \beta_n e^{\beta_n h_1}]D^{-1}Y = -e^{h_1}$. Note that

$$\begin{aligned} & [\beta_1 e^{\beta_1 h_1}, ..., \beta_n e^{\beta_n h_1}]D^{-1}Y \\ = & [\beta_1 e^{\beta_1 h_1}, ..., \beta_n e^{\beta_n h_1}] \begin{bmatrix} \frac{1}{\beta_1(1-\beta_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\beta_2(1-\beta_2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{\beta_{N+2}(1-\beta_{N+2})} \end{bmatrix} Y \\ = & det \begin{bmatrix} \frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2}h_1} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2}h_1} \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h_2} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2}h_2} \\ \frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1 h_1} + \frac{1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2}h_1} + \frac{1}{K_2}e^{\beta_{N+2}h_2}) \\ \frac{e^{\beta_1 h_1}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2}h_1}}{1-\beta_{N+2}} \end{bmatrix} \end{aligned}$$

Therefore

$$
\begin{aligned}
-e^{h_1} &= \frac{K_1}{det(A)}[\beta_1 e^{\beta_1 h_1}, ..., \beta_n e^{\beta_n h_1}]D^{-1}Y \\
&= det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2} h_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2} h_1} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h_2} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h_2} \\
\frac{1}{\beta_1}(e^{\beta_1 h_1}+\frac{K_1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(e^{\beta_{N+2} h_1}+\frac{K_1}{K_2}e^{\beta_{N+2} h_2}) \\
\frac{e^{\beta_1 h_1}}{1-\beta_1} & \cdots & \frac{e^{\beta_{N+2} h_1}}{1-\beta_{N+2}}
\end{bmatrix} \\
&\times det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-}e^{\beta_{N+2} h_1} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-}e^{\beta_{N+2} h_1} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} h_2} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 h_2} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} h_2} \\
\frac{1}{\beta_1}(\frac{1}{K_1}e^{\beta_1 h_1}+\frac{1}{K_2}e^{\beta_1 h_2}) & \cdots & \frac{1}{\beta_{N+2}}(\frac{1}{K_1}e^{\beta_{N+2} h_1}+\frac{1}{K_2}e^{\beta_{N+2} h_2}) \\
\frac{1}{\beta_1}e^{\beta_1 h_1} & \cdots & \frac{1}{\beta_{N+2}}e^{\beta_{N+2} h_1}
\end{bmatrix}^{-1} \\
&= det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{1}{1-\beta_1} & \cdots & \frac{1}{1-\beta_{N+2}}
\end{bmatrix} \\
&\times det\begin{bmatrix}
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N-}^-} \\
\frac{1}{\beta_1-\eta_1^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2} \Delta h} \\
\vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N+}^+}e^{\beta_1 \Delta h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N+}^+}e^{\beta_{N+2} \Delta h} \\
\frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 \Delta h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2} \Delta h}) \\
\frac{1}{\beta_1 K_1} & \cdots & \frac{1}{\beta_{N+2} K_1}
\end{bmatrix}^{-1} \\
&= -detA_1/detA_2
\end{aligned}
$$

which verifies (4.3). □

**Proof of Propositin 4.2.** Note that $detB(h) = 0$ if and only if $det\widehat{B}(h) = 0$ where

$$\widehat{B}(h) = \begin{bmatrix} \frac{1}{\beta_1-1}(1+e^{(\beta_1-1)h}) & \cdots & \frac{1}{\beta_{N+2}-1}(1+e^{(\beta_{N+2}-1)h}) \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1}(1+\frac{K_1}{K_2}e^{\beta_1 h}) & \cdots & \frac{1}{\beta_{N+2}}(1+\frac{K_1}{K_2}e^{\beta_{N+2}h}) \\ \frac{1}{\beta_1-\eta_1^+}e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+}e^{\beta_{N+2}h} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+}e^{\beta_1 h} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}e^{\beta_{N+2}h} \end{bmatrix}$$

As $h = 0$, $det\widehat{B}(0) = 2(1+\frac{K_1}{K_2})detZ^{(N+2)}$ where

$$Z^{(N+2)} = \begin{bmatrix} \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N+2}-1} \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^-}^-} \\ \frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N+2}} \\ \frac{1}{\beta_1-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+} \end{bmatrix}.$$

We show that $detZ^{(N+2)} > 0$. For simplicity, we set $\alpha_1 = 1$, $\alpha_{n+1} = \eta_n^-$ for $1 \le n \le N^-$, $\alpha_{N^-+2} = 0$, and $\alpha_{N^-+2+m} = \eta_m^+$ for $1 \le m \le N^+$. Then the entry $z_{i,j}^{(N+2)}$ of $Z^{(N+2)}$ is equal to $\frac{1}{\beta_j-\alpha_i}$. Note that for $2 \le i \le N+2$, $z_{i,j}^{(N+2)} > 0$ if $i \le j$ and $z_{i,j}^{(N+2)} < 0$ if $i > j$. For $2 \le k \le N+1$, let $Z^{(k)}$ be the $k \times k$ matrix with entries $z_{i,j}^{(k)} = z_{N+2-k+i,N+2-k+j}^{(N+2)} = \frac{1}{\beta_{N+2-k+j}-\alpha_{N+2-k+i}}$ for $1 \le i,j \le k$. First we show that $detZ^{(k)} > 0$ for $2 \le k \le N+1$. For $k = 2$, we have

$$\begin{aligned} detZ^{(2)} &= det\begin{bmatrix} \frac{1}{\beta_{N+1}-\alpha_{N+1}} & \frac{1}{\beta_{N+2}-\alpha_{N+1}} \\ \frac{1}{\beta_{N+1}-\alpha_{N+2}} & \frac{1}{\beta_{N+2}-\alpha_{N+2}} \end{bmatrix} \\ &= \frac{1}{(\beta_{N+1}-\alpha_{N+1})(\beta_{N+2}-\alpha_{N+2})} - \frac{1}{(\beta_{N+2}-\alpha_{N+1})(\beta_{N+1}-\alpha_{N+2})} \\ &= \frac{1}{(\beta_{N+1}-\eta_{N^+-1}^+)(\beta_{N+2}-\eta_{N^+}^+)} - \frac{1}{(\beta_{N+2}-\eta_{N^+-1}^+)(\beta_{N+1}-\eta_{N^+}^+)} > 0 \end{aligned}$$

Before proceeding, we need the fact that $detZ^{(k)} \neq 0$ for $3 \le k \le N+2$. Indeed, assume that $Z^{(k)}L = 0$ for some column vector $L = (l_1, \cdots, l_k)'$. Let

$$F_k(x) = \sum_{j=1}^{k} \frac{l_j}{\beta_{N+2-k+j}-x} = \frac{G_k(x)}{\prod_{n=1}^{k}(\beta_{N+2-k+n}-x)}, \tag{7.11}$$

where $G_k(x) = \sum_{j=1}^{k} l_j \prod_{n=1,n\neq j}^{k}(\beta_{N+2-k+n}-x)$ is a polynomial with $deg(G_k(x)) \le k-1$. Since

$$G_k(\alpha_{N+2-k+i}) = \prod_{n=1}^{k}(\beta_{N+2-k+n}-\alpha_{N+2-k+i})F_k(\alpha_{N+2-k+i}) = 0 \quad 1 \le i \le k.$$

$G_k(x)$ has at least $k$ distinct roots which implies $G_k(x) = 0$. Besides, from (7.11), we have

$$l_j = \frac{G_k(\beta_{N+2-k+j})}{\prod_{n=1,n\neq j}^{k}(\beta_{N+2-k+n}-\beta_{N+2-k+j})} = 0 \quad \text{for } 1 \le j \le k.$$

This implies $Z^{(k)}L = 0$ has no nontrivial solutions and hence $detZ^{(k)} \neq 0$. Suppose that $detZ^{(k)} > 0$ for some $2 \leq k \leq N$. Consider the system of equations $Z^{(k+1)}\widetilde{L}^{(k+1)} = e_1^{(k+1)}$ where $\widetilde{L}^{(k+1)}$ and $e_1^{(k+1)} \equiv [1, 0, \cdots, 0]'$ are $(k+1) \times 1$ column vectors. Let

$$\widetilde{F}_{k+1}(x) = \sum_{j=1}^{k+1} \frac{\widetilde{l}_j^{(k+1)}}{\beta_{N+2-(k+1)+j} - x} = \frac{\widetilde{G}_{k+1}(x)}{\prod_{n=1}^{k+1}(\beta_{N+2-k-1+n} - x)}.$$

Then $\widetilde{G}_{k+1}(x) = \sum_{j=1}^{k+1} \widetilde{l}_j^{(k+1)} \prod_{n=1, n\neq j}^{k+1}(\beta_{N+2-(k+1)+n} - x)$ is a polynomial with $deg(\widetilde{G}_{k+1}(x)) \leq k$ and

$$\widetilde{G}_{k+1}(\alpha_{N+2-(k+1)+i}) = \prod_{n=1}^{k}(\beta_{N+2-(k+1)+n} - \alpha_{N+2-(k+1)+i})\widetilde{F}_{k+1}(\alpha_{N+2-(k+1)+i}) = 0, \quad 2 \leq i \leq k+1.$$

Therefore, we have $\{x|\widetilde{G}_{k+1}(x) = 0\} = \{\alpha_{N+2-(k+1)+i}|2 \leq i \leq k+1\}$ which implies

$$\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+i})\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+i+1}) < 0 \quad \text{for } 1 \leq i \leq k-1.$$

Since

$$\begin{aligned}\widetilde{l}_j\widetilde{l}_{j+1} &= \frac{\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j})\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j+1})}{\prod_{n=1,n\neq j}^{k+1}(\beta_{N+2-(k+1)+n} - \beta_{N+2-(k+1)+j})\prod_{n=1,i\neq j+1}^{k+1}(\beta_{N+2-(k+1)+n} - \beta_{N+2-(k+1)+j+1})} \\ &= \frac{\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j})\widetilde{G}_{k+1}(\beta_{N+2-(k+1)+j+1})}{\prod_{n=1,n\neq j,j+1}^{k+1}(\beta_{N+2-(k+1)+n} - \beta_j)^2(\beta_{N+2-(k+1)+j} - \beta_{N+2-k+j+1})(\beta_{N+2-(k+1)+j+1} - \beta_{N+2-(k+1)+j})} \\ &> 0\end{aligned}$$

for $1 \leq j \leq k$, $\widetilde{l}_j$'s have the same sign. In addition, because the entries of the first row in $Z^{(k+1)}$ are positive and $\widetilde{F}_{k+1}(\alpha_{N+2-(k+1)+1}) = 1$, we obtain $\widetilde{l}_j > 0$ for all $1 \leq j \leq k+1$. On the other hand, by Cramer's rule, we know that $\widetilde{l}_1 = \frac{detZ^{(k)}}{detZ^{(k+1)}}$. Therefore, $detZ^{(k+1)} > 0$. Since $detZ^{(2)} > 0$, by induction, this implies $detZ^{(n)} > 0$ for $1 \leq n \leq N+1$. Consider the system of equations $Z^{(N+2)}\widetilde{L} = e_1$ where $\widetilde{L}$ and $e_1 = [1, 0, 0, ..., 0]'$ are $(N+2) \times 1$ column vectors. Let $\widetilde{F}_{N+2}(x) = \sum_{j=1}^{N+2} \frac{\widetilde{l}_j}{\beta_j - x} = \frac{\widetilde{G}_{N+2}(x)}{\prod_{n=1}^{N+2}(\beta_n - x)}$. Then $\widetilde{G}_{N+2}(x) = \sum_{j=1}^{N+2} \widetilde{l}_j \prod_{n=1,n\neq j}^{N+2}(\beta_n - x)$ is a polynomial with $deg(\widetilde{G}_{N+2}) \leq N+1$. By the equation $Z^{(N+2)}\widetilde{L} = e_1$, we have $\widetilde{G}_{N+2}(\alpha_n) = 0$ for $2 \leq n \leq N+2$. By similar arguments as above, we know that the entries of $\widetilde{L}$ have the same sign. By Lemma 3.1, $F_{N+2}(x)$ is well-defined on $[0,1]$ and in addition, $F_{N+2}(x) \in C([0,1]) \cap C^1(0,1)$. Besides, $Z^{(N+2)}\widetilde{L} = e_1$ implies $\widetilde{F}_{N+2}(1) = 1$ and $\widetilde{F}_{N+2}(0) = 0$. Therefore, by the mean value theorem, there exists $x_0 \in (0,1)$ such that $1 = \widetilde{F}(1) - \widetilde{F}(0) = \widetilde{F}'(x_0)(1-0) = \sum_{i=1}^{N+2} \frac{\widetilde{l}_i}{(\beta_i - x_0)^2}$ which implies that the entries of $\widetilde{L}$ are positive. On the other hand, by Cramer's rule, we have $\widetilde{l}_1 = \frac{detZ^{(N+1)}}{detZ^{(N+2)}}$. Therefore, $detZ^{(N+2)} > 0$.

Consider the determinant $detW$ where

$$W = \begin{bmatrix} \frac{1}{\beta_1 - 1} & \cdots & \frac{1}{\beta_{N^-+1} - 1} & 0 & \cdots & 0 \\ \frac{1}{\beta_1 - \eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1} - \eta_1^-} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\beta_1 - \eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1} - \eta_{N^-}^-} & 0 & \cdots & 0 \\ \frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N^-+1}} & \frac{1}{\beta_{N^-+2}} & \cdots & \frac{1}{\beta_{N+2}} \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2} - \eta_1^+} & \cdots & \frac{1}{\beta_{N+2} - \eta_1^+} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2} - \eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2} - \eta_{N^+}^+} \end{bmatrix}$$

Let

$$W^{(1)} = \begin{bmatrix} \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N^-+1}-1} \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} \end{bmatrix} \quad \text{and } W^{(2)} = \begin{bmatrix} \frac{1}{\beta_{N^-+2}} & \cdots & \frac{1}{\beta_{N+2}} \\ \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+} \end{bmatrix}.$$

By similar arguments for the proof of the invertibility of $Z^{(k)}$, we observe that $W^{(1)}$ and $W^{(2)}$ are invertible. Therefore, $\left\{[\frac{1}{\beta_1-1},\cdots,\frac{1}{\beta_{N^-+1}-1}],[\frac{1}{\beta_1-\eta_1^-},\cdots,\frac{1}{\beta_{N^-+1}-\eta_1^-}],\cdots,[\frac{1}{\beta_1-\eta_{N^-}^-},\cdots,\frac{1}{\beta_{N^-+1}-\eta_{N^-}^-}]\right\}$ is a basis of $\mathbb{R}^{N^-+1}$ and hence,

$$detW = det \begin{bmatrix} \frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N^-+1}-1} & 0 & \cdots & 0 \\ \frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}} & \cdots & \frac{1}{\beta_{N+2}} \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+} \end{bmatrix} = detW^{(1)}detW^{(2)}.$$

We show that $detW < 0$. Consider the system of equations $W^{(1)}\widehat{L} = e_1$ where $\widehat{L}$ and $e^1 = [1,0,0,...,0]'$ are $(N^- + 1) \times 1$ column vectors. By the same arguments for $detZ^{(N+1)}$, we observe that the entries of $\widehat{L}$ have the same sign and $det\widetilde{W}^{(1)} > 0$ where

$$\widetilde{W}^{(1)} = \begin{bmatrix} \frac{1}{\beta_2-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\beta_2-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} \end{bmatrix}.$$

Because $w_{1,j}^{(1)} < 0$ for all $1 \le j \le N^- + 1$ and $\sum_{j=1}^{N^-+1} \frac{\widehat{l}_j}{\beta_j - 1} = 1$, the entries of $\widehat{L}$ are negative. On the other hand, by Cramer's rule, we have $\widehat{l}_1 = \frac{det\widetilde{W}^{(1)}}{detW^{(1)}}$ and hence, $detW^{(1)} < 0$. By a similar argument, we obtain $detW^{(2)} > 0$ and hence $detW < 0$. Since

$$
\begin{aligned}
&\lim_{h\to\infty} \frac{det\widehat{B}(h)}{e^{\sum_{i=N^-+2}^{N+2}\beta_i h}} \\
&= \lim_{h\to\infty} det \begin{bmatrix}
\frac{1+e^{(\beta_1-1)h}}{\beta_1-1} & \cdots & \frac{1+e^{(\beta_{N^-+1}-1)h}}{\beta_{N^-+1}-1} & \frac{e^{-\beta_{N^-+2}h}+e^{-h}}{\beta_{N^-+2}-1} & \cdots & \frac{e^{-\beta_{N+2}h}+e^{-h}}{\beta_{N+2}-1} \\
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} & \frac{e^{-\beta_{N^-+2}h}}{\beta_{N^-+2}-\eta_1^-} & \cdots & \frac{e^{-\beta_{N+2}h}}{\beta_{N+2}-\eta_1^-} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} & \frac{e^{-\beta_{N^-+2}h}}{\beta_{N^-+2}-\eta_{N^-}^-} & \cdots & \frac{e^{-\beta_{N+2}h}}{\beta_{N+2}-\eta_{N^-}^-} \\
\frac{1+\frac{K_1}{K_2}e^{\beta_1 h}}{\beta_1} & \cdots & \frac{1+\frac{K_1}{K_2}e^{\beta_{N^-+1}h}}{\beta_{N^-+1}} & \frac{e^{-\beta_{N^-+2}h}+\frac{K_1}{K_2}}{\beta_{N^-+2}} & \cdots & \frac{e^{-\beta_{N+2}h}+\frac{K_1}{K_2}}{\beta_{N+2}} \\
\frac{e^{\beta_1 h}}{\beta_1-\eta_1^+} & \cdots & \frac{e^{\beta_{N^-+1}h}}{\beta_{N^-+1}-\eta_1^+} & \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{e^{\beta_1 h}}{\beta_1-\eta_{N^+}^+} & \cdots & \frac{e^{\beta_{N^-+1}h}}{\beta_{N^-+1}-\eta_{N^+}^+} & \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}
\end{bmatrix} \\
&= det \begin{bmatrix}
\frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N^-+1}-1} & 0 & \cdots & 0 \\
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} & 0 & \cdots & 0 \\
\frac{1}{\beta_1} & \cdots & \frac{1}{\beta_{N^-+1}} & \frac{K_1}{K_2}\frac{1}{\beta_{N^-+2}} & \cdots & \frac{K_1}{K_2}\frac{1}{\beta_{N+2}} \\
0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}
\end{bmatrix} \\
&= det \begin{bmatrix}
\frac{1}{\beta_1-1} & \cdots & \frac{1}{\beta_{N^-+1}-1} & 0 & \cdots & 0 \\
\frac{1}{\beta_1-\eta_1^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_1^-} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\beta_1-\eta_{N^-}^-} & \cdots & \frac{1}{\beta_{N^-+1}-\eta_{N^-}^-} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{K_1}{K_2}\frac{1}{\beta_{N^-+2}} & \cdots & \frac{K_1}{K_2}\frac{1}{\beta_{N+2}} \\
0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_1^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_1^+} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{\beta_{N^-+2}-\eta_{N^+}^+} & \cdots & \frac{1}{\beta_{N+2}-\eta_{N^+}^+}
\end{bmatrix} \\
&= \frac{K_1}{K_2} detW < 0,
\end{aligned}
$$

we observe that for sufficiently large $h$, $det\widehat{B}(h) < 0$. In addition, we have $det\widehat{B}(0) = 2(1 + \frac{K_1}{K_2})detZ^{(N+2)} > 0$. By the intermediate value theorem, this implies that $det\widehat{B}(h) = 0$ has a positive solution $\Delta h$ and hence we complete the proof. □

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