Journal of Computational Physics 230 (2011) 4377–4383

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Short note

A note on pressure accuracy in immersed boundary method for Stokes flow

Kuan-Yu Chen [a], Ko-An Feng [b], Yongsam Kim [c], Ming-Chih Lai [d,*]

[a] Department of Applied Mathematics, National Chiao Tung University, 1001, Ta Hsueh Road, Hsinchu 30010, Taiwan
[b] National Center for Theoretical Sciences, National Tsing Hua University, No. 101, Sec. 2, Kuang Fu Road, Hsinchu 30043, Taiwan
[c] Department of Mathematics, Chung-Ang University, Dongjak-Gu, Heukseok-Dong 221, Seoul 156756, South Korea
[d] Center of Mathematical Modeling and Scientific Computing and Department of Applied Mathematics, National Chiao Tung University, 1001, Ta Hsueh Road, Hsinchu 30010, Taiwan

ARTICLE INFO

Article history:
Received 16 November 2010
Received in revised form 15 February 2011
Accepted 9 March 2011
Available online 16 March 2011

Keywords:
Immersed boundary method
Pressure accuracy
Discrete delta function
Stokes flow

ABSTRACT

In this short note, we provide a simplified one-dimensional analysis and two-dimensional numerical experiments to predict that the overall accuracy for the pressure or indicator function in immersed boundary calculations is first-order accurate in L_1 norm, half-order accurate in L_2 norm, but has $O(1)$ error in L_∞ norm. Despite the pressure has $O(1)$ error near the interface, the velocity field still has the first-order convergence in immersed boundary calculations.

1. Introduction

The immersed boundary (IB) method was first proposed by Peskin in [8] as a computational tool to study the blood flow in the heart. Over the past few decades, the IB method has become a useful computational framework for solving fluid–structure interaction problems, see a recent review [9]. Whereas the IB method has an impressive ease of the implementations, it is well-known that the method is only first-order accurate since it is a smoothing method rather than the sharp capturing method. Recently, a rigorous convergence proof of the velocity for Stokes flow in the immersed boundary formulation has been provided by Mori [7]. The author has proved that the velocity is roughly first-order accurate in the L_∞ norm; however, it has no conclusion about the accuracy of the pressure.

In [2], Beyer and LeVeque have analyzed the accuracy of one-dimensional model for the immersed boundary method. They gave some insight about the accuracy of solving one-dimensional heat equation with a delta source term by choosing appropriate discrete delta functions. Tornberg and Engquist [11] used the regularization technique to analyze the numerical accuracy of some elliptic PDEs. They have verified numerically that, for two-dimensional Poisson equation with singular delta source term, the standard centered difference approximation with smoothing discrete delta function is first-order accurate in L_∞ norm and second-order accurate in L_1 norm. They also showed that, away from the interface, the scheme has a better accuracy which is expected in the case of smooth problems without singular source term. Another more accurate approach for solving PDEs with singular sources is to incorporate the solution or its derivative jumps across the interface into the finite difference scheme such as the immersed interface method [6]. Readers who are interested in IIM can refer to a recent survey book by Li and Ito [5].

* Corresponding author. Tel.: +886 3 5731361; fax: +886 3 5724679.
E-mail address: mclai@math.nctu.edu.tw (M.-C. Lai).

0021-9991/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2011.03.019

In this paper, we shall provide a simplified one-dimensional analysis and two-dimensional numerical experiments to predict that the pressure in immersed boundary calculations is first-order accurate in L_1 norm, half-order accurate in L_2 norm, but has $O(1)$ error in L_∞ norm. Notice that, it has no surprise that the pressure has $O(1)$ error near the interface, since it is discontinuous across the interface, and a smoothing method such as the immersed boundary method will not be able to capture the right discontinuity. A rigorous convergence proof for the two-dimensional problems will need a further investigation in the future.

Since the pressure appears in its gradient form in Stokes equations, one might wonder how this $O(1)$ point-wise error near the interface affects the overall accuracy of the velocity field. The pressure gradient and the immersed boundary force terms have the same singular behavior (a delta function singularity), finding the velocity field involves solving Poisson equation with a singular delta function source. It is known that the numerical approximation of Poisson equation with a singular delta function source only has first-order accuracy in L_∞ norm [11]. Even though the calculated pressure has $O(1)$ error near the interface, its gradient has the same discretization error as the singular delta function force near the interface. Thus, the overall accuracy of velocity field in IB method is first-order accurate in L_∞ norm.

2. Model problems

The problem which we are interested in arises from solving the stationary Stokes flow in a two-dimensional domain Ω with one-dimensional boundary (or interface) Γ immersed in Ω as

$$-\nabla p + \mu \Delta \mathbf{v} + \int_\Gamma \mathbf{F}(s)\,\delta^2(\mathbf{x} - \mathbf{X}(s))\,ds = 0, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{2}$$

The immersed boundary force $\mathbf{F}(s) = (F^x(s), F^y(s))$ is only exerted along the interface Γ so that the integral is along the one-dimensional interface Γ while the Dirac delta function $\delta^2(\mathbf{x}) = \delta(x)\delta(y)$ is two-dimensional. Thus, the above immersed boundary formulation is a typical singular problem with a delta function source. By taking the divergence operator on Eq. (1) and using the incompressibility constraint (2), we obtain the pressure equation

$$\Delta p(\mathbf{x}) = \nabla \cdot \int_\Gamma \mathbf{F}(s)\,\delta^2(\mathbf{x} - \mathbf{X}(s))\,ds. \tag{3}$$

Notice that, this equation is a Poisson equation with a source term which involves derivatives of the Dirac delta function. Once pressure is found, we can find the velocity by solving Eq. (1) which results two Poisson equations for the velocity components. In the immersed boundary computations, the pressure equation often uses periodic or Neumann boundary condition; however, throughout this paper, we simply use the Dirichlet boundary condition since we are more concerned with the accuracy caused by the derivatives of Dirac delta function near the interface.

Another way of solving Stokes equations (1) and (2) is to discretize them directly and solve the resultant linear system for the velocity and pressure simultaneously. This procedure is mostly adopted by the problem with periodic boundary conditions in which FFT can be conveniently applied or fast Stokes solver is available. It seems that the above two solution procedures are different; however, they share the same accuracy behavior for the pressure. The reason is that the pressure in IB formulation is discontinuous across the interface and both solution procedures involve the derivatives of the pressure and singular terms. Therefore, the pressure accuracy predicted in the paper does not change despite different solution procedures.

Another example leading to the same type of equation as Eq. (3) appears when we use the idea of Unverdi and Tryggvason [10] to find the indicator function which is needed to track the regions of two-phase flow. If the viscosity is discontinuous across the immersed boundary, it can be represented by the following:

$$\mu(\mathbf{x}) = \mu_{\text{out}} + (\mu_{\text{in}} - \mu_{\text{out}}) I(\mathbf{x}),$$

where μ_{in} and μ_{out} are the viscosity inside and outside the interface, respectively. Since the indicator function has the value one ($I = 1$) inside the immersed boundary Γ and the value zero ($I = 0$) outside, it can be calculated as the following procedure [10]. Let Ω_{in} represent the interior region and $\mathbf{n}$ be the unit outward normal vector to the interface, then the indicator function can be represented by

$$I(\mathbf{x}) = \int_{\Omega_{\text{in}}} \delta^2(\mathbf{x} - \widetilde{\mathbf{x}})\,d\widetilde{\mathbf{x}}.$$

By taking the gradient and then divergence operators on both sides, we have

$$\nabla I(\mathbf{x}) = -\int_\Gamma \mathbf{n}\,\delta^2(\mathbf{x} - \mathbf{X}(s))\,ds,$$

$$\Delta I(\mathbf{x}) = -\nabla \cdot \int_\Gamma \mathbf{n}\,\delta^2(\mathbf{x} - \mathbf{X}(s))\,ds. \tag{4}$$

Thus, the indicator function can be obtained by solving the same type of Poisson equation as Eq. (3) with the special singular forcing term $\mathbf{F}(s) = -\mathbf{n}(s)$. In this paper, our goal is to consider the standard finite difference scheme with smoothing discrete delta function to solve the Eqs. (3) and (4), and to investigate its numerical accuracy.

3. One-dimensional analysis

Let us consider the one-dimensional counterpart as

$$\frac{d^2u}{dx^2} = c\,\frac{d}{dx}\delta(x-\alpha), \quad 0 \leqslant x \leqslant 1, \tag{5}$$

with boundary conditions $u(0) = u(1) = 0$, and the immersed boundary point is located at $x = \alpha \in (0,1)$. As is known, the exact solution $u(x)$ can be represented as

$$u(x) = \int_0^1 G(x;y)\,c\,\frac{d}{dy}\delta(y-\alpha)dy, \tag{6}$$

where $G(x;y)$ is the well-known Green's function defined as $G_{xx}(x;y) = \delta(x-y)$, and can be explicitly written as

$$G(x;y) = \begin{cases} x(y-1), & 0 \leqslant x \leqslant y, \\ y(x-1), & y < x \leqslant 1. \end{cases}$$

For convenience, we set $c = 1$. By formally applying the integration by parts to Eq. (6), we obtain

$$u(x) = -\int_0^1 \frac{d}{dy}G(x;y)\delta(y-\alpha)dy. \tag{7}$$

Now we specify a uniform grid with grid points $x_j = jh$, $j = 0,1,\ldots,N$ with $h = 1/N$, and then discretize the Eq. (5) with $c = 1$ by the standard centered difference scheme

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = \frac{\delta_h(x_{j+1}-\alpha) - \delta_h(x_{j-1}-\alpha)}{2h}, \tag{8}$$

where δ_h is the discrete delta function [8] defined as

$$\delta_h(x) = \begin{cases} \frac{1}{4h}\left(1+\cos(\frac{\pi x}{2h})\right), & \text{if } |x| \leqslant 2h, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

This discrete delta function satisfies

$$\sum_{m=0}^{m=N} \delta_h(x_m - \alpha)h = 1, \tag{10}$$

which is the corresponding basic requirement for the delta function. Other discrete delta functions can be found in [2,12]; however, the usage of other delta functions leads to the same conclusion as will be given in this paper. For clarity, we denote the first-order and second-order centered difference operators as D_h and D_h^2, respectively. Similar to the analytic solution in Eq. (6), the discrete solution U_j of Eq. (8) can also be written as

$$U_j = h\sum_{m=0}^{N} G_{jm} D_h \delta_h(x_m - \alpha), \tag{11}$$

where $G_{jm} = G(x_j, x_m)$ is the discrete version of Green's function defined as

$$G_{jm} = \begin{cases} x_j(x_m-1), & 0 \leqslant j \leqslant m, \\ x_m(x_j-1), & m < j \leqslant N. \end{cases}$$

We can immediately check that G satisfies $D_h^2 G_{jm} = \frac{1}{h}\delta_{jm}$ where δ_{jm} is the Kronecker delta function.

Now, by taking summation by parts and the property of the discrete delta function, the numerical solution U_j can be rewritten as

$$U_j = -h\sum_{m=0}^{N} D_h G_{jm} \delta_h(x_m - \alpha). \tag{12}$$

Using the similar approach as in [11], for any discrete point x_j, $j = 0,1,\ldots,N$, the point-wise error between U_j and $u(x_j)$ can be represented by

$$|U_j - u(x_j)| = \left|h\sum_{m=0}^{N} D_hG_{jm}\delta_h(x_m-\alpha) - \int_0^1 \frac{d}{dy}G(x_j;y)\delta(y-\alpha)dy\right| \leqslant \left|h\sum_{m=0}^{N} D_hG_{jm}\delta_h(x_m-\alpha)\right.$$
$$\left.-h\sum_{m=0}^{N}\frac{d}{dy}G(x_j;y)\Big|_{y=x_m}\delta_h(x_m-\alpha)\right| + \left|h\sum_{m=0}^{N}\frac{d}{dy}G(x_j;y)\Big|_{y=x_m}\delta_h(x_m-\alpha) - \frac{d}{dy}G(x_j;y)\Big|_{y=\alpha}\right| = E_1 + E_2.$$

Using the fact that the derivative of Green's function is

$$\frac{d}{dy}G(x_j;y)\Big|_{y=x_m} = \begin{cases} x_j, & 0 \leqslant x_j \leqslant x_m, \\ x_j - 1, & x_m < x_j \leqslant 1 \end{cases} \tag{13}$$

and that its discrete counterpart is

$$D_hG_{jm} = \begin{cases} x_j, & j < m, \\ x_j - \frac{1}{2}, & j = m, \\ x_j - 1, & j > m, \end{cases} \tag{14}$$

we can conclude that the first part of the error E_1 becomes

$$E_1 = \left|h\frac{1}{2}\delta_h(x_j-\alpha)\right| = \begin{cases} O(1), & \text{when } |x_j - \alpha| \leqslant 2h, \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

The second part of the error E_2 is simply an interpolating error for the function $\frac{d}{dy}G(x_j;y)\Big|_{y=x_m}$. Using the formula in Eq. (13) and the first moment condition in (10), since the discrete delta function has finite support 4h, we can obtain

$$E_2 = \begin{cases} \left|h\sum_{m=0}^{j-1}\delta_h(x_m-\alpha)\right|, & x_j \leqslant \alpha, \\ \left|h\sum_{m=j}^{N}\delta_h(x_m-\alpha)\right|, & x_j > \alpha, \end{cases}$$

which implies that

$$E_2 = \begin{cases} O(1), & \text{as } |x_j-\alpha| \leqslant 2h, \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

From the above analysis, one can immediately see that the point-wise error appears only at some points around the singular point α, which means that the maximum error $\|u_h - u\|_\infty$ is of order $O(1)$. For the same reason, we can conclude that L_1 ($\|u_h - u\|_1$) and L_2 ($\|u_h - u\|_2$) errors are of order $O(h)$ and $O(h^{1/2})$, respectively. Our numerical results in next section will confirm this conclusion.

4. Numerical results

4.1. One-dimensional problem

In this subsection, we consider the following one-dimensional problem:

$$\frac{d^2u}{dx^2} = c\,\frac{d}{dx}\delta(x-\alpha) + g, \quad 0 < x < 1, \tag{17}$$

with the interface at the point $\alpha = \pi/6$. The exact solution is given as

$$u(x) = \begin{cases} x^3 + 2\alpha x^2, & \text{if } x \leqslant \alpha, \\ 7(x^3-1)/3, & \text{if } x > \alpha, \end{cases} \tag{18}$$

where the jump size c of the solution u at the interface is set to be $-(2\alpha^3 + 7)/3$. The regular source term g can be easily computed by the analytic solution.

Throughout this section, the ratio between two consecutive errors is computed as $\frac{\|u-u_h\|}{\|u-u_{h/2}\|}$, where u is the exact solution and u_h is the numerical solution with the mesh width $h = 1/N$. As the mesh is refined, the asymptotic ratios of 1, 1.414, 2 represent that the corresponding orders of accuracy are zeroth-order ($\log_2 1$), half-order $\left(\log_2\sqrt{2}\right)$ and first-order ($\log_2 2$).

Table 1 shows the order of accuracy for our test which verifies exactly our one-dimensional analysis in the previous section. Notice that, when we implemented different versions of discrete delta function given in [2,12], we observed the same behavior of the errors.

Table 1
Order of accuracy for one-dimensional test.

N	$\|u-u_h\|_\infty$	Ratio	$\|u-u_h\|_2$	Ratio	$\|u-u_h\|_1$	Ratio
32	8.8827E−01	–	1.7057E−01	–	4.2479E−02	–
64	6.1709E−01	1.4394	1.0736E−01	1.5887	1.9004E−02	2.2352
128	1.1847E−00	0.5208	1.0708E−01	1.0026	1.2980E−02	1.4641
256	1.1579E−00	1.0231	7.4120E−02	1.4446	6.3791E−03	2.0347
512	1.1030E−00	1.0497	5.0171E−02	1.4773	3.0741E−03	2.0750

4.2. Two-dimensional problems

For two-dimensional problem, we generally write the equation as

$$\Delta u = \nabla \cdot \int_\Gamma \mathbf{F}(s)\, \delta^2(x - \mathbf{X}(s))\, ds + g \tag{19}$$

in a domain $\Omega = [a,b] \times [c,d]$ with Dirichlet boundary conditions. We first divide the domain Ω into $M \times N$ uniform grids with mesh width $\Delta x = \Delta y = h$, and $u_{i,j}$ denotes the discrete solution at (x_i, y_j) where $x_i = ih$, $i = 0,1,\ldots,M$ and $y_j = jh$, $j = 0,1,\ldots,N$. We also choose a collection of marker points $\mathbf{X}(s_k) = (X_k, Y_k)$ along the interface Γ with the mesh points $s_k = k\Delta s$. Here, the marker mesh width Δs is about a half of h. Then we use the standard centered difference scheme to discrete Eq. (19) as

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = \frac{f^x_{i+1/2,j} - f^x_{i-1/2,j}}{h} + \frac{f^y_{i,j+1/2} - f^y_{i,j-1/2}}{h} + g_{i,j},$$

where

$$f^x_{i+1/2,j} = \sum_k F^x(s_k)\, \delta_h(x_i + h/2 - X_k)\delta_h(y_j - Y_k)\, \Delta s,$$
$$f^y_{i,j+1/2} = \sum_k F^y(s_k)\, \delta_h(x_i - X_k)\delta_h(y_j + h/2 - Y_k)\, \Delta s$$

and $f^x_{i-1/2,j}$, $f^y_{i,j-1/2}$ are defined in a similar fashion. The resultant matrix equation can be solved efficiently by the fast direct solver in Fishpack [1].

Example 1. For the first example, we test the accuracy of the indicator function which is obtained by solving Eq. (4). For completeness, we test three different interface Γ in the domain $[-1,1] \times [-1,1]$ as follows.

1. Γ is a circle centered at (0,0) with the radius 0.3.
2. Γ is an ellipse centered at (0,0) with the major radius 0.9 and minor radius 0.1.
3. Γ is a simple closed curve written in polar coordinates: $r = 0.5 + 0.25\cos(5\theta)$.

Tables 2–4 show the errors and convergence ratios for these three different cases. One can immediately see that, whereas the indicator function does indeed have an $O(1)$ error in maximum norm, it is first-order convergent in L_1 norm and half-order convergent in L_2 norm. The results are consistent to the one-dimensional analysis.

Example 2. In this example, we test an analytic solution which arises from the pressure Eq. (3) in Stokes flow developed in [3]. This example is also used in [4] for a simple version of immersed interface method. The computational domain is $\Omega = [-2,2] \times [-2,2]$, and the interface is a unit circle centered at (0,0), i.e., $\mathbf{X}(\theta) = (\cos\theta, \sin\theta)$. The exact solution is written in polar coordinates as

$$u(r,\theta) = \begin{cases} -r^3(\cos(3\theta) + \sin(3\theta)), & \text{if } r \leqslant 1, \\ -r^{-3}(\cos(3\theta) - \sin(3\theta)), & \text{if } r > 1 \end{cases}$$

Table 2
Convergent test for indicator function in case 1: a circle.

$M \times N$	$\|u-u_h\|_\infty$	Ratio	$\|u-u_h\|_2$	Ratio	$\|u-u_h\|_1$	Ratio
32 × 32	3.6463E−01	–	1.3162E−01	–	6.3848E−02	–
64 × 64	4.5555E−01	0.8004	9.5529E−02	1.3777	3.2182E−02	1.9839
128 × 128	4.8736E−01	0.9347	7.2764E−02	1.3128	1.6837E−02	1.9113
256 × 256	4.8610E−01	1.0026	4.9738E−02	1.4629	8.2361E−03	2.0443
512 × 512	4.9805E−01	0.9759	3.4744E−02	1.4315	4.0955E−03	2.0109

Table 3
Convergent test for indicator function in case 2: an ellipse.

$M \times N$	$\|u-u_h\|_\infty$	Ratio	$\|u-u_h\|_2$	Ratio	$\|u-u_h\|_1$	Ratio
32×32	6.7302E−01	–	1.9248E−01	–	1.2276E−01	–
64×64	5.0021E−01	1.3454	1.4391E−01	1.3374	6.5139E−02	1.8845
128×128	4.9922E−01	1.0019	9.7919E−02	1.4697	3.1954E−02	2.0385
256×256	4.9834E−01	1.0017	6.8903E−02	1.4211	1.5951E−02	2.0033
512×512	4.9617E−01	1.0043	4.8685E−02	1.4152	7.9510E−03	2.0061

Table 4
Convergent test for indicator function in case 3: a simple closed curve.

$M \times N$	$\|u-u_h\|_\infty$	Ratio	$\|u-u_h\|_2$	Ratio	$\|u-u_h\|_1$	Ratio
32×32	5.9986E−01	–	2.5162E−01	–	2.0860E−01	–
64×64	5.5492E−01	1.0810	1.8259E−01	1.3780	1.0827E−01	1.9266
128×128	5.3029E−01	1.0464	1.2910E−01	1.4143	5.4431E−02	1.9891
256×256	5.1669E−01	1.0263	9.0547E−02	1.4257	2.7064E−02	2.0111
512×512	5.1194E−01	1.0092	6.4251E−02	1.4092	1.3547E−02	1.9977

Table 5
Convergent test for Example 2.

$M \times N$	$\|u-u_h\|_\infty$	Ratio	$\|u-u_h\|_2$	Ratio	$\|u-u_h\|_1$	Ratio
32×32	1.5643E−00	–	9.5960E−01	–	2.0421E−00	–
64×64	1.7182E−00	0.9104	6.9177E−01	1.3871	1.1867E−00	1.7209
128×128	1.8342E−00	0.9367	4.9447E−01	1.3990	6.4868E−01	1.8292
256×256	1.9086E−00	0.9610	3.4999E−01	1.4128	3.4044E−01	1.9054
512×512	1.9284E−00	0.9897	2.4775E−01	1.4126	1.7495E−01	1.9459

Table 6
Convergent test for Example 3.

$M \times N$	$\|u-u_h\|_\infty$	Ratio	$\|u-u_h\|_2$	Ratio	$\|u-u_h\|_1$	Ratio
32×32	5.9986E−01	–	2.5162E−01	–	2.0860E−01	–
64×64	5.5492E−01	1.0809	1.8259E−01	1.3780	1.0827E−01	1.9266
128×128	5.3029E−01	1.0464	1.2910E−01	1.4143	5.4431E−01	1.9891
256×256	5.1669E−01	1.0263	9.0547E−02	1.4257	2.7064E−02	2.0111
512×512	5.1194E−01	1.0092	6.4251E−02	1.4092	1.3547E−02	1.9977

and the boundary force $\mathbf{F}(\theta) = 2\sin(3\theta)(\mathbf{X}'(\theta) + \mathbf{X}(\theta))$. Table 5 shows the convergence ratios which again verifies our one-dimensional analysis.

Example 3. Finally we consider Eq. (19) for which the domain is $\Omega = [-1,1] \times [-1,1]$ and the interface Γ is described as a simple closed curve $r = 0.5 + 0.25\cos(5\theta)$ in polar coordinates. The analytic solution u is given by

$$u = \begin{cases} (x^2-1)(y^2-1)+1, & \text{if } (x,y) \in \Omega_{\text{out}}, \\ (x^2-1)(y^2-1), & \text{if } (x,y) \in \Omega_{\text{in}}, \end{cases}$$

where Ω_{out} and Ω_{in} represent the exterior and interior regions, respectively. Note that the boundary force $\mathbf{F}$ is simply the normal vector $\mathbf{n}$ along the interface Γ. The external source g can be easily computed from the exact solution u.

Table 6 shows the convergence tests of the numerical solutions for Example 3. The results are consistent to what we expect, i.e. they are first-order accurate in L_1 norm, half-order accurate in L_2 norm, but have $O(1)$ errors in L_∞ norm.

Acknowledgment

Y. Kim is supported by the National Research Foundation ofKorea, Grant 2010-0006165. M.-C. Lai is supported in part by National Science Council of Taiwan under research Grant NSC-97-2628-M-009-007-MY3, NSC-98-2115-M-009-014-MY3, and the support of NCTS in Taiwan.

References

[1] J. Adams, P. Swarztrauber, R. Sweet, Fishpack – A package of Fortran subprograms for the solution of separable elliptic partial differential equations. Available from: <http://www.netlib.org/fishpack>.
[2] R.P. Beyer, R.J. LeVeque, Analysis of a one-dimensional model for the immersed boundary method, SIAM J. Numer. Anal. 29 (1992) 332–364.
[3] R. Cortez, The method of regularized Stokeslets, SIAM J. Sci. Comput. 23 (2001) 1204–1225.
[4] M.-C. Lai, H.-C. Tseng, A simple implementation of the immersed interface methods for Stokes flows with singular forces, Comput. Fluids 37 (2008) 99–106.
[5] Z. Li, K. Ito, The Immersed Interface Method, SIAM, 2006.
[6] R.J. LeVeque, Z. Li, The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, SIAM J. Numer. Anal. 31 (1994) 1019–1044.
[7] Y. Mori, Convergence proof of the velocity field for a Stokes flow immersed boundary method, Commun. Pure Appl. Math. LXI (2008) 1213–1263.
[8] C.S. Peskin, Numerical analysis of blood flow in the heart, J. Comput. Phys. 25 (1977) 220–252.
[9] C.S. Peskin, The immersed boundary method, Acta Numerica 11 (2002) 479–517.
[10] S.O. Unverdi, G. Tryggvason, A front-tracking method for viscous incompressible multi-fluid flows, J. Comput. Phys. 100 (1992) 25–37.
[11] A.-K. Tornberg, B. Engquist, Numerical approximations of singular source terms in differential equations, J. Comput. Phys. 200 (2004) 462–488.
[12] X. Yang, X. Zhang, Z. Li, G.-W. He, A Smoothing technique for discrete delta functions with application to immersed boundary method in moving boundary simulations, J. Comput. Phys. 228 (2009) 7821–7836.