On optimal stopping problems for matrix-exponential Lévy processes

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Abstract

In this paper, we consider the optimal stopping problems for a general class of reward functions under jump-diffusion processes. Given an American call-type reward function, following the averageing problem approach(see, for example, Alili and Kyprianou [1], Kyprianou and Surya [8], Novikov and Shiryaev [13], and Surya [15]), we give an explicit formula for solutions of the corresponding average problem. Based on this explicit formula, we obtain the optimal level and the value function for the American call-type optimal problems.

Keywords: Optimal stopping problem; American call-type reward functions; Averaging problem; Lévy process and matrix-exponential distribution

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1 Introduction

The optimal stopping problems we consider in this paper will be of the form

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) \tag{1.1}$$

where $X = \{X_t : t \geq 0\}$ under $\mathbb{P}_x$ is a Lévy process started from $X_0 = x$. Further, g is a continuous reward function, $r > 0$ and $\mathcal{T}$ is a family of stopping times with respect to the natural filtration generated by X, $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. The optimal stopping problem consists of finding the optimal stopping time τ^* such that $V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$. Also we need to find the corresponding optimal reward (the value function).

The general theory of optimal stopping rules for Markov processes says that the value function $V(x)$ is the smallest r-excessive majorant of $g(x)$ (i.e., the smallest function $V(x)$ such that $V(x) \geq g(x), V(x) \geq e^{-rt}\mathbb{E}_x(V(X_t))$ for all $x \in \mathbb{R}$ and $t \geq 0$, and $\lim_{t \downarrow 0} e^{-rt}\mathbb{E}_x(V(X_t)) = V(x)$). Moreover, the first entry time τ_D of the process X into the stopping region $D = \{x \in \mathbb{R}, V(x) = g(x)\}$ is the smallest optimal stopping time(if it exists) and hence the value function is given by the formula $\mathbb{E}_x(e^{-r\tau_D} g(X_{\tau_D}))$. (See, for example, Shiryayev [14] Lemma 3 p.118 and Theorem 1 p.124.)

For diffusion processes, many authors in the literature obtained that the boundaries of the stopping region D is determined by using the smooth pasting condition for the value function and solving the optimal stopping problem is reduced to solving a corresponding Stefan's free boundary problem. Unfortunately, once the problem (1.1) is driven by Lévy processes, the smooth pasting condition may break down. For example, Alili and Kyprianou [1] considered the problem of pricing the American put option under a Lévy process. They showed that the smooth pasting conditions

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hold if and only if 0 is regular for $(-\infty, 0)$ for the Lévy process X. Similar results were also obtained by Kyprianou and Surya [8] for the integer power reward function g. Recently, inspired by the works of Boyarchenko and Levendorskiĭ [5], Surya [15] proposed an averaging problem approach for solving the optimal stopping problem (1.1) in a general setting. (For earlier works on this approach, see, for example, Alili and Kyprianou [1], Kyprianou and Surya [8] and Novikov and Shiryaev [13].) The averaging problem approach does not appeal to a free boundary problem associated to the optimal stopping problem. Given a reward function, in terms of solutions for the corresponding averaging problem, we have a fluctuation identity for overshoots for a Lévy process. With this identity, Surya [15] showed by martingale arguments that an optimal solution can be founded if the solutions to the averaging problem have certain monotonicity properties. Use these approaches, he was able to recover many known results in literature(see, for example, [1], [8], [10] and [13]).

In this paper, we consider a jump-diffusion process X of the form in (2.2) below. Under this model assumption, we give an explicit formula in Theorem 3.3 for solutions of the averaging problem for a general class of reward function. (Our result depends on the recent work of Lewis and Mordecki in [9].) Moreover, when the reward function g is of American call-type, we derive sufficient criterions on the explicit solutions for the corresponding averageing problem and g for the existence of optimal boundary. In particular, we can evaluate the optimal boundaries for the boundary value problem (1.1) by solving an equation and then obtain an explicit formula for the value functions. (For details, see Theorem 3.7.)

The outline of the paper is as follows. In Section 2 we recall some main results of Surya [15] and [9]. When the reward function g is in a special class of functions, denoted by π_0, we present in Section 3 an explicit formula Q_g for the solutions of the corresponding averaging problem and obtain sufficient conditions on g and Q_g for the optimality. The special class of reward functions contains many known examples in literature. In particular we verify that if the reward function g is a sufficiently regular function, Q_g is consistent with that in Surya [15]. To illustrate our results, we resolve the optimal stopping problem (1.1) in Section 4 for some specific reward functions. Our results are consistent with that of Kyprianou and Surya [8], Mordecki [10], and Boyarchenko and Levendorskii [5]. Also it is worth noting that, under our model assumption, our examples in Section 4 show that the sufficient conditions for optimality in Theorem 3.7 are easier to verify than that in the literature.

2 Preliminaries

2.1 Sufficient conditions for optimality

A Lévy process is a real-valued process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ satisfying the following properties:

(a) The paths of X are almost surely right continuous with left limits.

(b) $X_0 = 0$ almost surely.

(c) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

(d) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

A Lévy process starts from $X_0 = x$ is simply defined as $x + X_t$ for $t \geq 0$ and we denote its law by $\mathbb{P}_x$. For every Lévy process, we have $\mathbb{E}(e^{\mathrm{i}uX_t}) = e^{t\psi(u)}$, where ψ is called the characteristic exponent of X and is given by the formula

$$\psi(u) = \mathrm{i}au - \frac{1}{2}b^2u^2 + \int_{\mathbb{R}} (e^{\mathrm{i}ux} - 1 - \mathrm{i}ux \cdot 1_{\{|x|<1\}})\pi(dx).$$

Here $a \in \mathbb{R}$, $b \geq 0$ and π is a measure on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\pi(dx) < \infty$.

Denote by e_r an exponential random variable with parameter $r > 0$, independent of the process X and denote by

$$M_r = \sup_{0 \leq s \leq \mathrm{e}_r} X_s \quad \text{and} \quad I_r = \inf_{0 \leq s \leq \mathrm{e}_r} X_s,$$

the supremum and the infimum of the Lévy process X killed at the independent exponential random time e_r. It is well known that $X_{\mathrm{e}_r}-I_r$ and I_r are independent, and $X_{\mathrm{e}_r}-I_r$ has the same distribution as M_r. From these, we obtain the following Wiener-Hopf factorization formula,

$$\mathbb{E}e^{\mathrm{i}uX_{\mathrm{e}_r}} = \frac{r}{r-\psi(u)} = \psi_r^+(u)\psi_r^-(u),$$

where $\psi_r^+(u) = \mathbb{E}(e^{\mathrm{i}uM_r})$ and $\psi_r^-(u) = \mathbb{E}(e^{\mathrm{i}uI_r})$.

To solve the optimal stopping problem (1.1), as in Kyprianou and Surya [8], Novikov and Shiryaev [13], Surya [15] and many others, we first introduce the corresponding averaging problem. We say that the reward function g is of American call-type if g is a non-decreasing continuous function and $\{g > 0\} = (\widehat{a}, \infty)$ for some $\widehat{a} < \infty$. Given a reward function g of American call-type and $r > 0$, the averaging problem for the optimal stopping problem (1.1) consists of finding a function $\widetilde{Q}_g$ satisfying the condition

$$\mathbb{E}\left(\widetilde{Q}_g(x + M_r)\right) = g(x) \tag{2.1}$$

for every $x > \widehat{a}$. Following similar arguments(with some necessary modifications) as in Surya [15], we have the following sufficient conditions for optimality.

Theorem 2.1 *Given a reward function g of American call-type and set $H := \{g > 0\} = (\widehat{a}, \infty)$ for some $\widehat{a} < \infty$. Suppose that $\widetilde{Q}_g$ is a continuous function on H that solves the averaging problem (2.1) for every $x \in H$. We assume further that there exists $\widehat{x} \in H$ such that $\widetilde{Q}_g(\widehat{x}) = 0$, $\widetilde{Q}_g(x)$ is non-decreasing for $x > \widehat{x}$ and $\widetilde{Q}_g(x) \le 0$ for $\widehat{a} < x < \widehat{x}$. Then the solution to the optimal stopping problem (1.1) is given by*

$$V(x) = \sup_{\tau\in M} \mathbb{E}_x(e^{-r\tau}g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})),$$

where x^* is the largest root of $\widetilde{Q}_g(x) = 0$ in $(\widehat{a}, \infty)$ and $\tau^* = \inf\{t > 0 : X_t > x^*\}$. Moreover, we have for every $x \in \mathbb{R}$,

$$V(x) = \mathbb{E}\left(\widetilde{Q}_g(x + M_r)1_{\{x+M_r>x^*\}}\right).$$

Remark 2.2 *(a) By replacing X with its dual process $\widehat{X} = -X$ and x with $-x$, similar results were also obtained in Surya [15] for American put-type optimal stopping problems.*
(b) It is worth noting that if $g(x) = \mathbb{E}\left[\int_0^\infty e^{-rt}h(x + X_t)dt\right]$ for some bounded function h, then by the Wiener-Hopf fluctuation identity, the function $\widetilde{Q}_g(x) = \mathbb{E}_x[h(I_r)]$ is a solution of the averageing problem (2.1). From this observation and the above theorem, we recover parts of Theorem 2 in Deligiannidis et al. [7]. In fact, similar results were obtained earlier by Boyarchenko and Levendorskiǐ [6] when the reward function g is of call-like or put-like.

2.2 Matrix-exponential Lévy processes

Recall that a distribution F on $(0, \infty)$ with a density function f is called a **matrix-exponential** distribution if its Laplace transform is a rational function. The two dense classes of distributions on $(0, \infty)$, phase-type distributions and generalized hyperexponential distributions, are both subclasses of matrix-exponential distributions. Also it is easy to show that F is a matrix-exponential distribution if, and only if, its density function is of the form

$$f(y) = \sum_{j=1}^{M} R_j(y)e^{-b_j y}, \qquad y > 0,$$

where $M \ge 1$, each R_j is a polynomial in y, and $b_j \in \mathbb{C}$ are distinct and $\mathcal{R}e\, b_j > 0$. (For details, see Asmussen and Albrecher [2] or Asmussen [3].)

From now on, we consider the jump-diffusion process X of the form

$$X_t = X_0 + at + bW_t + \sum_{n=1}^{N_t^\lambda} Y_n + \sum_{k=1}^{N_t^\mu} Z_k, \qquad t \geq 0. \tag{2.2}$$

Here, $a \in \mathbb{R}\backslash\{0\}$, $b \geq 0$, $W = (W_t, t \geq 0)$ is a standard Brownian motion, $N^\lambda = (N_t^\lambda; t \geq 0)$, as well as $N^\mu = (N_t^\mu; t \geq 0)$ are the Poisson processes with rate $\lambda > 0$ and $\mu > 0$, respectively. Also, $Y = (Y_n, n \in \mathbb{N})$ and $Z = (Z_k, k \in \mathbb{N})$ are sequences of independent random variables with identical matrix-exponential distributions given by,

$$dF^{(+)}(x) = p_1(x)dx = (1_{\{x>0\}} \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{c_{kj}\beta_k^j x^{j-1}}{(j-1)!} e^{-\beta_k x})dx$$

and

$$dF^{(-)}(x) = p_2(x)dx = (1_{\{x<0\}} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{\widetilde{c}_{pm}\alpha_p^m (-x)^{m-1}}{(m-1)!} e^{\alpha_p x})dx,$$

respectively. Here, the parameters $c_{kj}, \beta_k, \widetilde{c}_{pm}$, and α_p can in principle take complex values, but if we order α_p and β_k by their real parts then α_1 and β_1 must be real, while the others may be complex with $0 < \beta_1 < \mathcal{R}e\,(\beta_2) \leq \cdots \leq \mathcal{R}e\,(\beta_{v_1})$ and $0 < \alpha_1 < \mathcal{R}e\,(\alpha_2) \leq \cdots \leq \mathcal{R}e\,(\alpha_{v_2})$. The random variables W, N^λ, N^μ, Y and Z are assumed to be independent. Note that the characteristic exponent of X is given by

$$\psi(z) = \mathrm{i}az - \frac{b^2z^2}{2} + \lambda\Big[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k} c_{kj}\Big(\frac{\mathrm{i}\beta_k}{z+\mathrm{i}\beta_k}\Big)^j - 1\Big] + \mu\Big[\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p} \widetilde{c}_{pm}\Big(\frac{-\mathrm{i}\alpha_p}{z-\mathrm{i}\alpha_p}\Big)^m - 1\Big]. \tag{2.3}$$

We quote the following results of Lewis and Mordecki [9].

Theorem 2.3 *(a)The equation $r - \psi(z) = 0$ has, in the half-plane $\mathcal{I}m\,(z) < 0$, μ_1 distinct roots $-\mathrm{i}\rho_1, \cdots, -\mathrm{i}\rho_{\mu_1}$(with respective multiplicities $m_1, ..., m_{\mu_1}$), ordered such that $0 < \mathcal{R}e\,(\rho_1) \leq \mathcal{R}e\,(\rho_2) \leq \cdots \leq \mathcal{R}e\,(\rho_{\mu_1})$. The root $-\mathrm{i}\rho_1$ is purely imaginary. Furthermore, the total root count $\sum_{j=1}^{\mu_1} m_j$ is equal to $\overline{n}$ when $-X^-$ is a subordinator and $\overline{n}+1$ when $-X^-$ is not a subordinator. Here, $\overline{n} = \sum_{k=1}^{v_1} n_k$. On the other hand, the equation $r - \psi(z) = 0$ has, in the half-plane $\mathcal{I}m\,(z) > 0$, μ_2 distinct roots $-\mathrm{i}\widetilde{\rho}_1, \cdots, -\mathrm{i}\widetilde{\rho}_{\mu_2}$(with respective multiplicities $\widetilde{m}_1, ..., \widetilde{m}_{\mu_2}$), ordered such that $\mathcal{R}e\,(\widetilde{\rho}_{\mu_2}) \leq \cdots \leq \mathcal{R}e\,(\widetilde{\rho}_1) < 0$. The root $-\mathrm{i}\widetilde{\rho}_1$ is purely imaginary. Furthermore, the total root count $\sum_{j=1}^{\mu_2} \widetilde{m}_j$ is equal to $\overline{\ell}$ when X^+ is a subordinator and $\overline{\ell}+1$ when X^+ is not a subordinator. Here, $\overline{\ell} = \sum_{p=1}^{v_2} \ell_p$.*
(b)The minimum I_r has rational Laplace transform ψ_r^- given by

$$\psi_r^-(u) = \prod_{k=1}^{v_2}\left(\frac{u-\mathrm{i}\alpha_k}{-\mathrm{i}\alpha_k}\right)^{\ell_k} \prod_{j=1}^{\mu_2}\left(\frac{\mathrm{i}\widetilde{\rho}_j}{u+\mathrm{i}\widetilde{\rho}_j}\right)^{\widetilde{m}_j}$$

and the maximum M_r has rational Laplace transform ψ_r^+ given by

$$\psi_r^+(u) = \prod_{k=1}^{v_1}\left(\frac{u+\mathrm{i}\beta_k}{\mathrm{i}\beta_k}\right)^{n_k} \prod_{j=1}^{\mu_1}\left(\frac{\mathrm{i}\rho_j}{u+\mathrm{i}\rho_j}\right)^{m_j}.$$

3 Main results

Recall that $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form in (2.2) and $-\mathrm{i}\widetilde{\rho}_1, \cdots, -\mathrm{i}\widetilde{\rho}_{\mu_2}, -\mathrm{i}\rho_1, \cdots, -\mathrm{i}\rho_{\mu_1}$ are distinct roots of $r - \psi(z) = 0$ with $\mathcal{R}e\,(\widetilde{\rho}_{\mu_2}) \leq \cdots \leq \mathcal{R}e\,(\widetilde{\rho}_1) \leq 0 < \mathcal{R}e\,(\rho_1) \leq \mathcal{R}e\,(\rho_2) \leq \cdots \leq \mathcal{R}e\,(\rho_{\mu_1})$. We assume further that $m_1 = m_2 = \cdots = m_{\mu_1} = \widetilde{m}_1 = \cdots = \widetilde{m}_{\mu_2} = 1$. Under these assumptions, our goal is to solve the optimal stopping problem (1.1) for a given continuous reward function g and $r > 0$. To do this, we first give an explicit formula for solutions of the corresponding averaging problem (2.1). In terms of the explicit solutions for the averaging problem and the given reward function, we find sufficient conditions for the existence of the optimal stopping time for the problem (1.1).

First we observe, by Theorem 2.3, that the distribution of $\inf_{0\leq s\leq e_r} X_s$ is given by

$$\mathbb{P}\left(\inf_{0\leq s\leq e_r} X_s \in dy\right) = 1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\delta_0(dy) + 1_{\{\mu_2\geq 1\}}\left[(\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta e^{-\widetilde{\rho}_\eta y})1_{\{y<0\}}dy\right] \tag{3.1}$$

where

$$\widetilde{d}_0 = \prod_{j=1}^{\mu_2}(-\widetilde{\rho}_j)\prod_{k=1}^{v_2}\alpha_k^{-\ell_k} \tag{3.2}$$

and

$$\widetilde{d}_j = (-1)\prod_{k=1}^{v_2}\left(\frac{\widetilde{\rho}_j+\alpha_k}{\alpha_k}\right)^{\ell_k}\prod_{m=1,m\neq j}^{\mu_2}\left(\frac{\widetilde{\rho}_m}{-\widetilde{\rho}_j+\widetilde{\rho}_m}\right), \text{ for } 1\leq j\leq \mu_2. \tag{3.3}$$

Also the distribution of $\sup_{0\leq s\leq e_r} X_s$ is given by

$$\mathbb{P}\left(\sup_{0\leq s\leq e_r} X_s \in dy\right) = 1_{\{a<0 \text{ and } b=0\}}d_0\delta_0(dy) + 1_{\{\mu_1\geq 1\}}\left[(\sum_{j=1}^{\mu_1}d_j\rho_j e^{-\rho_j y})1_{\{y>0\}}dy\right] \tag{3.4}$$

where

$$d_0 = \prod_{j=1}^{\mu_1}\rho_j\prod_{k=1}^{v_1}\beta_k^{-n_k} \tag{3.5}$$

and

$$d_k = \prod_{j=1}^{v_1}\left(\frac{\beta_j-\rho_k}{\beta_j}\right)^{n_j}\prod_{i=1,i\neq k}^{\mu_1}\left(\frac{\rho_i}{\rho_i-\rho_k}\right), \text{ for } 1\leq k\leq \mu_1. \tag{3.6}$$

In the following, we follow the convention that $\prod_{i=1,i\neq k}^{\mu_1} = 1$ (resp., $\prod_{m=1,m\neq j}^{\mu_2} = 1$) in the case $\mu_1 = 1$ (resp., $\mu_2 = 1$). We also need the following facts.

Lemma 3.1 *Suppose $\mu_1 \geq 1$ and $\mu_2 \geq 1$. Then*

(a) For $i = 1, .., \mu_1$, we have

$$-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_i)^j} - \mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\rho_i)^m} + (\mu+\lambda+r) - a\rho_i - \frac{b^2\rho_i^2}{2} = 0, \tag{3.7}$$

and for $\xi = 1, ..., \mu_2$, we obtain

$$-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\xi)^j} - \mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\widetilde{\rho}_\xi)^m} + (\mu+\lambda+r) - a\widetilde{\rho}_\xi - \frac{b^2\widetilde{\rho}_\xi^{\,2}}{2} = 0. \tag{3.8}$$

(b) For $1 \leq k \leq v_1$ and $1 \leq \xi \leq n_k$, we get

$$\sum_{j=1}^{\mu_1}\frac{d_j\rho_j}{(\beta_k-\rho_j)^\xi} = \begin{cases} 1_{\{\xi=1\}}\cdot d_0, & \textit{if } a<0 \textit{ and } b=0,\\ 0, & \textit{otherwise.}\end{cases} \tag{3.9}$$

(c) For $1 \leq p \leq v_2$ and $1 \leq m \leq \ell_p$, we acquire

$$\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j}{(\alpha_p+\widetilde{\rho}_j)^m} = \begin{cases} 1_{\{m=1\}}\widetilde{d}_0, & \textit{if } a>0 \textit{ and } b=0,\\ 0, & \textit{otherwise.}\end{cases} \tag{3.10}$$

(d) For any complex numbers $A_{p,m}$ and ω, we have

$$\sum_{\eta=1}^{\mu_2}\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{A_{p,m}\widetilde{d}_\eta\widetilde{\rho}_\eta}{\omega-\widetilde{\rho}_\eta}\left(\frac{1}{(\alpha_p+\widetilde{\rho}_\eta)^m}-\frac{1}{(\alpha_p+\omega)^m}\right)=\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{A_{p,m}}{(\alpha_p+\omega)^m}\left(\widetilde{d}_0 1_{\{a>0 \text{ and } b=0\}}\right)\tag{3.11}$$

(e) For any complex numbers A_s, we obtain

$$\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s A_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\rho_s-\widetilde{\rho}_\eta)}\left[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}-a\rho_s-\frac{b^2\rho_s^2}{2}-\left(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}-a\widetilde{\rho}_\eta-\frac{b^2\widetilde{\rho}_\eta^2}{2}\right)\right]$$

$$=1_{\{a>0 \text{ and } b=0\}}\sum_{s=1}^{\mu_1}\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(-\widetilde{d}_0)d_s\rho_s A_s\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\rho_s)^m}.\tag{3.12}$$

(f) The following identities hold:

$$\begin{cases}\dfrac{-ad_0}{r}\displaystyle\sum_{j=1}^{\mu_2}\widetilde{d}_j\widetilde{\rho}_j=1, & \text{if } a<0 \text{ and } b=0\\[2ex] \dfrac{a\widetilde{d}_0}{r}\displaystyle\sum_{j=1}^{\mu_1}d_j\rho_j=1, & \text{if } a>0 \text{ and } b=0\\[2ex] \dfrac{1}{r}\displaystyle\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta(\frac{b^2}{2})=1, & \text{if } b\neq 0.\end{cases}\tag{3.13}$$

Proof. (a) The identities in part (a) follow from the facts that $-\mathrm{i}\widetilde{\rho}_1,\cdots,-\mathrm{i}\widetilde{\rho}_{\mu_2}$ and $-\mathrm{i}\rho_1,\cdots,-\mathrm{i}\rho_{\mu_1}$ are solutions of $r-\psi(z)=0$ and (2.3).

(b) Since $m_j=1$ for $j=1,...,\mu_1$, by Theorem 2.3, we obtain that

$$\psi_r^+(u)=\prod_{k=1}^{v_1}\left(\frac{u+\mathrm{i}\beta_k}{\mathrm{i}\beta_k}\right)^{n_k}\prod_{j=1}^{\mu_1}\left(\frac{\mathrm{i}\rho_j}{u+\mathrm{i}\rho_j}\right)\tag{3.14}$$

Note that if $a<0$ and $b=0$ then $\mu_1=\sum_{k=1}^{v_1}n_k$; otherwise, $\mu_1=\sum_{k=1}^{v_1}n_k+1$. Applying fractional decomposition to the right hand side of (3.14) gives

$$\prod_{k=1}^{v_1}\left(\frac{u+\mathrm{i}\beta_k}{\mathrm{i}\beta_k}\right)^{n_k}\prod_{j=1}^{\mu_1}\left(\frac{\mathrm{i}\rho_j}{u+\mathrm{i}\rho_j}\right)=1_{\{\mu_1\geq 1\}}\sum_{j=1}^{\mu_1}\frac{d_j\rho_j\mathrm{i}}{u+\mathrm{i}\rho_j}+1_{\{a<0 \text{ and } b=0\}}d_0\tag{3.15}$$

where $\lim_{u\to-\mathrm{i}\rho_j}(u+\mathrm{i}\rho_j)\psi_r^+(u)=d_j\rho_j\mathrm{i}$ and d_j are given by (3.6). For $1\leq k\leq v_1,$ and $1\leq\xi\leq n_k$, our results follow by differentiating both sides of (3.15) $(\xi-1)$-times at $u=-\mathrm{i}\beta_k$.

(c) Note that we have

$$\psi_r^-(u)=\prod_{k=1}^{v_2}\left(\frac{u-\mathrm{i}\alpha_k}{-\mathrm{i}\alpha_k}\right)^{\ell_k}\prod_{j=1}^{\mu_2}\left(\frac{\mathrm{i}\widetilde{\rho}_j}{u+\mathrm{i}\widetilde{\rho}_j}\right).\tag{3.16}$$

Also if $a>0$ and $b=0$ then $\mu_2=\sum_{k=1}^{v_2}\ell_k$; otherwise, $\mu_2=\sum_{k=1}^{v_2}\ell_k+1$. Applying fractional decomposition to the right hand side of (3.16) gives

$$\prod_{k=1}^{v_2}\left(\frac{u-\mathrm{i}\alpha_k}{-\mathrm{i}\alpha_k}\right)^{\ell_k}\prod_{j=1}^{\mu_2}\left(\frac{\mathrm{i}\widetilde{\rho}_j}{u+\mathrm{i}\widetilde{\rho}_j}\right)=1_{\{\mu_2\geq 1\}}\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j(-\mathrm{i})}{u+\mathrm{i}\widetilde{\rho}_j}+1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\tag{3.17}$$

where $\lim_{u\to -\mathrm{i}\widetilde{\rho}_j}(u+\mathrm{i}\widetilde{\rho}_j)\psi_r^-(u) = (-\mathrm{i})\widetilde{d}_j\widetilde{\rho}_j$ and $\widetilde{d}_j$ is given by (3.3). For $1 \le p \le v_2$ and $1 \le m \le \ell_p$, our result follows by differentiating both sides of (3.17) $(m-1)$-times at $u = \mathrm{i}\alpha_k$.

(d) Observe that $\sum_{q=1}^m \frac{1}{(\alpha_p+\widetilde{\rho}_\eta)^q(\alpha_p+\omega)^{m-q+1}} = \frac{1}{\omega-\widetilde{\rho}_\eta}\left(\frac{1}{(\alpha_p+\widetilde{\rho}_\eta)^m} - \frac{1}{(\alpha_p+\omega)^m}\right)$. Given any complex numbers $A_{p,m}$ and w, we have

$$\begin{aligned}
\sum_{\eta=1}^{\mu_2}\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{A_{p,m}\widetilde{d}_\eta\widetilde{\rho}_\eta}{\omega-\widetilde{\rho}_\eta}\left(\frac{1}{(\alpha_p+\widetilde{\rho}_\eta)^m} - \frac{1}{(\alpha_p+\omega)^m}\right) &= \sum_{\eta=1}^{\mu_2}\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\sum_{q=1}^{m}\frac{A_{p,m}\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\alpha_p+\widetilde{\rho}_\eta)^q(\alpha+\omega)^{m-q+1}} \\
&= \sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\sum_{q=1}^{m}\frac{A_{p,m}}{(\alpha_p+\omega)^{m-q+1}}\left(\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\alpha_p+\widetilde{\rho}_\eta)^q}\right) \\
&= \sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{A_{p,m}}{(\alpha_p+\omega)^m}\left(\widetilde{d}_0 1_{\{a>0 \text{ and } b=0\}}\right),
\end{aligned}$$

where the last equality follows from (3.10). The proof of (d) is complete.

(e) It is clear from (3.7) and (3.8) that

$$\begin{aligned}
\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s A_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\rho_s-\widetilde{\rho}_\eta)} \times \Bigg[&-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j} - \mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\rho_s)^m} - a\rho_s - \frac{b^2\rho_s^2}{2} \\
&- \left(-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j} - \mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\widetilde{\rho}_\eta)^m} - a\widetilde{\rho}_\eta - \frac{b^2\widetilde{\rho}_\eta^2}{2}\right)\Bigg] = 0.
\end{aligned}$$

This together with (3.11) yields (3.12).

(f) From Theorem 2.3, we have the following results.

$$\begin{cases}
\mu_1 = \sum_{p=1}^{v_1} n_p \text{ and } \mu_2 = \sum_{m=1}^{v_2}\ell_m + 1, & \text{if } a<0 \text{ and } b=0 \\
\mu_1 = \sum_{p=1}^{v_1} n_p + 1 \text{ and } \mu_2 = \sum_{m=1}^{v_2}\ell_m, & \text{if } a>0 \text{ and } b=0 \\
\mu_1 = \sum_{p=1}^{v_1} n_p + 1 \text{ and } \mu_2 = \sum_{m=1}^{v_2}\ell_m + 1, & \text{if } b\neq 0.
\end{cases} \tag{3.18}$$

By applying the Wiener-Hopf factorization formula and combining with (3.15) and (3.17), we see that for $b=0$,

$$\begin{aligned}
\frac{r}{r-\psi(u)} &= \frac{r\prod_{p=1}^{v_1}(u+\mathrm{i}\beta_p)^{n_p}\prod_{m=1}^{v_2}(u-\mathrm{i}\alpha_m)^{\ell_m}}{(-\mathrm{i}a)\prod_{j=1}^{\mu_1}(u+\mathrm{i}\rho_j)\prod_{k=1}^{\mu_2}(u+\mathrm{i}\widetilde{\rho}_k)} \\
&= \left(1_{\{\mu_1\ge 1\}}\sum_{j=1}^{\mu_1}\frac{d_j\rho_j\mathrm{i}}{u+\mathrm{i}\rho_j} + 1_{\{a<0 \text{ and } b=0\}}d_0\right)\left(1_{\{\mu_2\ge 1\}}\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j(-\mathrm{i})}{u+\mathrm{i}\widetilde{\rho}_j} + 1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\right),
\end{aligned} \tag{3.19}$$

and for $b\neq 0$,

$$\begin{aligned}
\frac{r}{r-\psi(u)} &= \frac{r\prod_{p=1}^{v_1}(u+\mathrm{i}\beta_p)^{n_p}\prod_{m=1}^{v_2}(u-\mathrm{i}\alpha_m)^{\ell_m}}{\frac{b^2}{2}\prod_{j=1}^{\mu_1}(u+\mathrm{i}\rho_j)\prod_{k=1}^{\mu_2}(u+\mathrm{i}\widetilde{\rho}_k)} \\
&= \left(\sum_{j=1}^{\mu_1}\frac{d_j\rho_j\mathrm{i}}{u+\mathrm{i}\rho_j}\right)\left(\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j(-\mathrm{i})}{u+\mathrm{i}\widetilde{\rho}_j}\right).
\end{aligned} \tag{3.20}$$

For the case $b = 0$, our results follow by multiplying both sides of (3.19) by u, letting $u \to \infty$ and using (3.18). For the case $b \neq 0$, we obtain our result by multiplying both sides of (3.20) by u^2, letting $u \to \infty$ and using (3.18). ■

Observe that if $v_1 \geq 1$ (i.e., there are upside jumps for the process X), then the function $\psi(\mathrm{i}z)$ is a real analytic function in $(-\beta_1, 0)$ with $\psi(0) = 0$ and $\lim_{z\downarrow -\beta_1} \psi(\mathrm{i}z) = \infty$. Hence, we have that $0 < \rho_1 < \beta_1$.

Definition 3.2 *We write $g \in \pi_0$ if the function $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every compact interval and there exist $A_1 > 0, A_2 > 0,$ and $\theta \in (0, \rho_1)$ such that $|g(x)| \leq A_1 + A_2 e^{\theta x}, \forall x \in \mathbb{R}$.*

For any $g \in \pi_0$, we define $Q_g(x)$ by the formula

$$
\begin{aligned}
Q_g(x) \quad &= \quad 1_{\{\mu_2 \geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} e^{\beta_k x} \int_x^\infty (y-x)^{j-\ell} g(y) e^{-\beta_k y} dy \\
&\qquad - \Big((a + \frac{b^2 \widetilde{\rho}_\eta}{2}) g(x) + \frac{b^2}{2} g'(x) \Big) \Bigg\} \\
&\qquad + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} e^{\beta_k x} \int_x^\infty (u-x)^{j-1} g(u) e^{-\beta_k u} du \\
&\qquad + (\lambda + \mu + r) g(x) - a g'(x) \Bigg\}.
\end{aligned}
\tag{3.21}
$$

We shows below that Q_g is a solution of the averaging problem (2.1).

Theorem 3.3 *For any $g \in \pi_0$ and $r > 0$,*

$$
E\Big[Q_g(M_r + x)\Big] = g(x), \forall x \in \mathbb{R}.
$$

Proof. Observe that

$$
\mathbb{E} Q_g(M_r + x) = \int_0^\infty Q_g(y+x) f_M(y) dy = \int_x^\infty Q_g(u) f_M(u-x) du \tag{3.22}
$$

where

$$
f_{M_r}(z) = 1_{\{a<0 \text{ and } b=0\}} d_0 \delta_0(dz) + 1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} d_j \rho_j e^{-\rho_j z} 1_{\{z>0\}}. \tag{3.23}
$$

We write

$$
Q_g(x) = 1_{\{\mu_2 \geq 1\}} \Big(Q_g^{(1)}(x) + Q_g^{(2)}(x) + Q_g^{(3)}(x) \Big) + 1_{\{a>0 \text{ and } b=0\}} \Big(Q_g^{(4)}(x) + Q_g^{(5)}(x) + Q_g^{(6)}(x) \Big), \tag{3.24}
$$

where

$$
Q_g^{(1)}(x) = \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} e^{\beta_k x} \int_x^\infty (y-x)^{j-\ell} g(y) e^{-\beta_k y} dy \Bigg\}, \tag{3.25}
$$

$$
Q_g^{(2)}(x) = \Bigg(- \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg) \frac{b^2}{2} g'(x), \tag{3.26}
$$

$$
Q_g^{(3)}(x) = - \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} (a + \frac{b^2 \widetilde{\rho}_\eta}{2}) g(x), \tag{3.27}
$$

$$Q_g^{(4)} = \frac{\widetilde{d}_0}{r}\left\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!}e^{\beta_k x}\int_x^\infty (u-x)^{j-1}g(u)e^{-\beta_k u}du\right\}, \tag{3.28}$$

$$Q_g^{(5)} = \frac{\widetilde{d}_0(-a)}{r}g'(x), \tag{3.29}$$

and

$$Q_g^{(6)} = \frac{\widetilde{d}_0(\lambda+\mu+r)}{r}g(x). \tag{3.30}$$

Taking account of $g \in \pi_0$ and using integration by parts, we obtain that

$$\begin{aligned}
& e^{\rho_s x}\int_x^\infty e^{(\beta_k-\rho_s)u}\int_u^\infty (y-u)^{j-\ell}g(y)e^{-\beta_k y}dydu \\
= & \sum_{\xi=1}^{j-\ell+1}\frac{-(j-\ell)!e^{\beta_k x}}{(j-\ell+1-\xi)!(\beta_k-\rho_s)^\xi}\int_x^\infty (y-x)^{j-\ell+1-\xi}g(y)e^{-\beta_k y}dy \\
& +\frac{(j-\ell)!e^{\rho_s x}}{(\beta_k-\rho_s)^{j-\ell+1}}\int_x^\infty g(y)e^{-\rho_s y}dy.
\end{aligned} \tag{3.31}$$

(For details, see the Appendix). For simplicity, we also write

$$I^{(1)}_{s,k,j-\ell} = e^{\rho_s x}\int_x^\infty e^{(\beta_k-\rho_s)u}\int_u^\infty (y-u)^{j-\ell}g(y)e^{-\beta_k y}dydu$$

and

$$I^{(2)}_{k,j-\ell+1-\xi} = e^{\beta_k x}\int_x^\infty (y-x)^{j-\ell+1-\xi}g(y)e^{-\beta_k y}dy.$$

Using (3.23), (3.25), and (3.31), we obtain for $\mu_2 \geq 1$

$$\begin{aligned}
& \int_x^\infty Q_g^{(1)}(u)f_M(u-x)du \\
= \; & 1_{\{\mu_1\geq 1\}}\frac{1}{r}\left\{\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta(-\lambda)(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell)!}\left(I^{(1)}_{s,k,j-\ell}\right)\right\} + 1_{\{a<0,b=0\}}d_0 Q_g^{(1)}(x) \\
= \; & 1_{\{\mu_1\geq 1\}}\frac{1}{r}\left\{\sum_{\eta=1}^{\mu_2}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\sum_{\xi=1}^{j-\ell+1}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell+1-\xi)!}\left(\sum_{s=1}^{\mu_1}\frac{d_s\rho_s}{(\beta_k-\rho_s)^\xi}\right)I^{(2)}_{k,j-\ell+1-\xi}\right. \\
& \left. +1_{\{\mu_1\geq 1\}}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\left(\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\rho_s)^{j-\ell+1}}\right)e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\right\} \\
& +1_{\{a<0 \text{ and } b=0\}}d_0 Q_g^{(1)}(x)
\end{aligned} \tag{3.32}$$

For $\mu_1 \geq 1$, it follows from (b) of Lemma 3.1 that for $k = 1, ..., v_1$, and $\xi = 1, ..., n_k$

$$\sum_{j=1}^{\mu_1}\frac{d_j\rho_j}{(\beta_k-\rho_j)^\xi} = \begin{cases} 1_{\{\xi=1\}}\cdot d_0, & \text{if } a<0 \text{ and } b=0 \\ 0, & \text{otherwise.}\end{cases}$$

This implies that for $\mu_2 \geq 1$

$$\begin{aligned}
& 1_{\{\mu_1\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\sum_{\xi=1}^{j-\ell+1}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell+1-\xi)!}\left(\sum_{s=1}^{\mu_1}\frac{d_s\rho_s}{(\beta_k-\rho_s)^\xi}\right)I^{(2)}_{k,j-\ell+1-\xi} \\
= \; & -1_{\{a<0 \text{ and } b=0\}}d_0 Q_g^{(1)}(x).
\end{aligned}$$

By this and the identity $\sum_{\ell=1}^{j} \frac{1}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\rho_s)^{j-\ell+1}} = \frac{1}{\rho_s-\widetilde{\rho}_\eta}\cdot\left(\frac{1}{(\beta_k-\rho_s)^j}-\frac{1}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)$, (3.32) becomes

$$\begin{aligned}&\int_x^\infty Q_g^{(1)}(u) f_M(u-x)du\\ =\;& 1_{\{\mu_1\ge1\}}\frac{1}{r}\left\{\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{\rho_s-\widetilde{\rho}_\eta}\left(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\right\}.\end{aligned} \tag{3.33}$$

Again, by using integration by parts together with $g\in\pi_0$, we get

$$e^{\rho_s x}\int_x^\infty g'(y)e^{-\rho_s y}dy = -g(x)+\rho_s e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy. \tag{3.34}$$

Combining (3.34) with (3.23) and (3.26) gives for $\mu_2\ge1$

$$\begin{aligned}&\int_x^\infty Q_g^{(2)}(u) f_M(u-x)du\\ =\;& 1_{\{\mu_1\ge1\}}(-\frac{b^2}{2r})\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2} d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\left(e^{\rho_s x}\int_x^\infty g'(y)e^{-\rho_s y}dy\right)\\ =\;& 1_{\{\mu_1\ge1\}}(-\frac{b^2}{2r})\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2} d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\left(-g(x)+\rho_s e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\right)\\ =\;& 1_{\{\mu_1\ge1\}}(\frac{b^2}{2r})g(x)\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2} d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta - 1_{\{\mu_1\ge1\}}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{\rho_s-\widetilde{\rho}_\eta}\left(\frac{\rho_s b^2}{2r}(\rho_s-\widetilde{\rho}_\eta)\right)e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy.\end{aligned} \tag{3.35}$$

Also, using (3.23) and (3.27), we have for $\mu_2\ge1$

$$\begin{aligned}&\int_x^\infty Q_g^{(3)}(u) f_M(u-x)du\\ =\;& 1_{\{\mu_1\ge1\}}\frac{(-1)}{r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2} d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta(a+\frac{b^2\widetilde{\rho}_\eta}{2})\left(e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\right)+1_{\{a<0,b=0\}}d_0 Q_g^{(3)}(x)\\ =\;& 1_{\{\mu_1\ge1\}}\frac{(-1)}{r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{\rho_s-\widetilde{\rho}_\eta}\left(a\rho_s+\frac{b^2\widetilde{\rho}_\eta\rho_s}{2}-(a\widetilde{\rho}_\eta+\frac{b^2\widetilde{\rho}_\eta^2}{2})\right)e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\\ &+1_{\{a<0 \text{ and } b=0\}}d_0 Q_g^{(3)}(x).\end{aligned} \tag{3.36}$$

Combining (3.33), (3.35) and (3.36), gives for $\mu_2\ge1$

$$\begin{aligned}&E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big]\\ =\;& 1_{\{\mu_1\ge1\}}\frac{1}{r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}\frac{d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\rho_s-\widetilde{\rho}_\eta)}e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\\ &\times\left[-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}-a\rho_s-\frac{b^2\rho_s^2}{2}-\left(-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}-a\widetilde{\rho}_\eta-\frac{b^2\widetilde{\rho}_\eta^2}{2}\right)\right]\\ &+1_{\{a<0 \text{ and } b=0\}}d_0 Q_g^{(3)}(x)+1_{\{\mu_1\ge1\}}(\frac{b^2}{2r})g(x)\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2} d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta.\end{aligned}$$

Using (e) of Lemma 3.1, we obtain that for $\mu_2 \geq 1$

$$\begin{aligned}
& E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
= \;& 1_{\{\mu_1\geq 1\}}1_{\{a>0 \text{ and } b=0\}}\sum_{s=1}^{\mu_1}\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(-\widetilde{d}_0)d_s\rho_s\mu(\alpha_p)^m\widetilde{c}_{pm}}{r(\alpha_p+\rho_s)^m}e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy \\
& +1_{\{a<0 \text{ and } b=0\}}d_0Q_g^{(3)}(x)+1_{\{\mu_1\geq 1\}}(\frac{b^2}{2r})g(x)\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta. \qquad (3.37)
\end{aligned}$$

Consequently, using (3.18), (3.13) and (3.24), we see that for the case $b\neq 0$,

$$\begin{aligned}
E\Big[Q_g(M_r+x)\Big] &= E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
&= g(x)\bigg(\frac{b^2}{2r}\sum_{s=1}^{\mu_1}\sum_{\eta=1}^{\mu_2}d_s\rho_s\widetilde{d}_\eta\widetilde{\rho}_\eta\bigg)=g(x),
\end{aligned}$$

and that for the case $a<0$ and $b=0$,

$$\begin{aligned}
E\Big[Q_g(M_r+x)\Big] &= E\Big[(Q_g^{(1)}+Q_g^{(2)}+Q_g^{(3)})(M_r+x)\Big] \\
&= g(x)\bigg(\frac{(-a)d_0}{r}\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta\bigg)=g(x).
\end{aligned}$$

It remains to consider the case $a>0$ and $b=0$. By (3.18), we have $\mu_1\geq 1$. It follows from (3.31), (3.23) and (3.28) that

$$\begin{aligned}
\int_x^\infty Q_g^{(4)}(u)f_M(u-x)du &= \sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\xi=1}^{j}\frac{\widetilde{d}_0\lambda(\beta_k)^j c_{kj}e^{\beta_k x}}{r(j-\xi)!}\bigg(\sum_{s=1}^{\mu_1}\frac{d_s\rho_s}{(\beta_k-\rho_s)^\xi}\bigg)\int_x^\infty(y-x)^{j-\xi}g(y)e^{-\beta_k y}dy \\
&\quad +\sum_{s=1}^{\mu_1}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{\widetilde{d}_0 d_s\rho_s(-\lambda)(\beta_k)^j c_{kj}e^{\rho_s x}}{r(\beta_k-\rho_s)^j}\int_x^\infty g(y)e^{-\rho_s y}dy \\
&= \sum_{s=1}^{\mu_1}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{\widetilde{d}_0 d_s\rho_s(-\lambda)(\beta_k)^j c_{kj}e^{\rho_s x}}{r(\beta_k-\rho_s)^j}\int_x^\infty g(y)e^{-\rho_s y}dy \qquad (3.38)
\end{aligned}$$

The last equality comes from (b) of Lemma 3.1. Using (3.34), (3.23), and (3.29), we have

$$\int_x^\infty Q_g^{(5)}(u)f_M(u-x)du=(\frac{-a\widetilde{d}_0}{r})\sum_{s=1}^{\mu_1}d_s\rho_s\bigg(-g(x)+\rho_s e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy\bigg). \qquad (3.39)$$

By (3.23) and (3.30), we have

$$\int_x^\infty Q_g^{(6)}(u)f_M(u-x)du=\frac{\widetilde{d}_0(\lambda+\mu+r)}{r}\sum_{s=1}^{\mu_1}d_s\rho_s e^{\rho_s x}\int_x^\infty g(y)e^{-\rho_s y}dy. \qquad (3.40)$$

It follows from (3.24) (3.37), (3.38), (3.39), and (3.40) that

$$\begin{aligned}
E\Big[Q_g(M_r+x)\Big] &= 1_{\{a>0 \text{ and } b=0\}}\sum_{s=1}^{\mu_1}\frac{\widetilde{d}_0 d_s\rho_s e^{\rho_s x}}{r}\int_x^\infty g(y)e^{-\rho_s y}dy \\
&\quad \times\bigg[-\lambda\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{(\beta_k)^j c_{kj}}{(\beta_k-\rho_s)^j}-\mu\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\rho_s)^m}+(\mu+\lambda+r)-a\rho_s\bigg] \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0 a}{r}(\sum_{j=1}^{\mu_1}d_j\rho_j)g(x)
\end{aligned}$$

By (a) and (f) of Lemma 3.1, we obtain $E\Big[Q_g(M_r+x)\Big] = g(x)$. The proof is complete. ■

We write $g \in \mathcal{R}$ if $g : \mathbb{R} \to \mathbb{R}$ is a L_1-integrable function such that the Fourier transform $\widehat{g}$, defind by $\widehat{g}(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega x} g(x)dx$ satisfies the integrability condition $\int_{-\infty}^{\infty}(1+|\omega|^3)|\widehat{g}(\omega)|d\omega < \infty$. As noted in Surya [15], the set $\mathcal{R}$ belongs to the class of $\mathcal{C}_b^3$. This implies that every element in $\mathcal{R}$ is also in π_0. In [15], Surya showed that if $g \in \mathcal{R}$, the function $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega x}\frac{\widehat{g}(\omega)}{\psi_r^-(\omega)}d\omega$ solves the American put-type averaging problem. In the following, we show that for $g \in \mathcal{R}$, the function $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega x}\frac{\widehat{g}(\omega)}{\psi_r^+(\omega)}d\omega$ coincides with the $Q_g(x)$ given by (3.21).

Proposition 3.4 *Given $g \in \mathcal{R}$ and define Q_g as in (3.21). Then we have*

$$Q_g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega x}\frac{\widehat{g}(\omega)}{\psi_r^+(\omega)}d\omega.$$

Proof. By the Fourier inversion formula, it is sufficient to prove that

$$\widehat{Q}_g(\omega) := \int_{-\infty}^{\infty} Q_g(y)e^{-\mathrm{i}\omega y}dy = \Big(\psi_r^+(\omega)\Big)^{-1}\widehat{g}(\omega). \tag{3.41}$$

As in (3.24), we write

$$Q_g(x) = 1_{\{\mu_2 \ge 1\}}\Big(Q_g^{(1)}(x) + Q_g^{(2)}(x) + Q_g^{(3)}(x)\Big) + 1_{\{a>0 \text{ and } b=0\}}\Big(Q_g^{(4)}(x) + Q_g^{(4)}(x) + Q_g^{(6)}(x)\Big). \tag{3.42}$$

To compute $\int_{-\infty}^{\infty} Q_g^{(1)}(y)e^{-\mathrm{i}\omega y}dy$, by using integration by parts, we have

$$\begin{aligned}
&\int_x^z e^{(\beta_k - \mathrm{i}\omega)y}\int_y^{\infty}(u-y)^{j-\ell}e^{-\beta_k u}g(u)dudy \\
= \;& \sum_{\xi=1}^{j-\ell+1}\frac{(j-\ell)!e^{(\beta_k-\mathrm{i}\omega)t}}{(j-\ell+1-\xi)!(\beta_k-\mathrm{i}\omega)^{\xi}}\int_t^{\infty}(y-t)^{j-\ell+1-\xi}e^{-\beta_k y}g(y)dy\Big|_{t=x}^{t=z} \\
&+\frac{(j-\ell)!}{(\beta_k-\mathrm{i}\omega)^{j-\ell+1}}\int_x^z g(y)e^{-\mathrm{i}\omega y}dy.
\end{aligned} \tag{3.43}$$

Here, for $1 \le \xi \le j-\ell+1$, we have

$$\lim_{x\to-\infty} e^{-\mathrm{i}\omega x}\int_x^{\infty}(y-x)^{j-\ell+1-\xi}e^{-\beta_k(y-x)}g(y)dy = 0 \tag{3.44}$$

and

$$\lim_{z\to\infty} e^{-\mathrm{i}\omega z}\int_z^{\infty}(y-z)^{j-\ell+1-\xi}e^{-\beta_k(y-z)}g(y)dy = 0. \tag{3.45}$$

To prove these, we observe that

$$\left|e^{-\mathrm{i}\omega t}\int_t^{\infty}(y-t)^{j-\ell+1-\xi}e^{-\beta_k(y-t)}g(y)dy\right| \le \int_0^{\infty}\left|u^{j-\ell+1-\xi}e^{-\beta_k u}g(u+t)\right|du. \tag{3.46}$$

Notice that $g \in \mathcal{R}$ implies $g \in \mathcal{C}_b^3$. Therefore g is uniformly continuous on $\mathbb{R}$. Combining this with $g \in L_1$ yields

$$\lim_{|x|\to\infty} g(x) = 0. \tag{3.47}$$

This together with the Dominated Convergence Theorem, implies that the right hand side of (3.46) converges to zero, as $t \to -\infty$ or $t \to \infty$. Therefore, we justify (3.44) and (3.45). From these, we observe for $\mu_2 \ge 1$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} Q_g^{(1)}(y)e^{-\mathrm{i}\omega y}dy \\
= &\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\left\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell (j-\ell)!}\int_{-\infty}^{\infty}e^{(\beta_k-\mathrm{i}\omega)y}\int_y^\infty (u-y)^j e^{-\beta_k u}g(u)dudy\right\} \\
= &\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\left\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\left(\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\mathrm{i}\omega)^{j-\ell+1}}\right)\int_{-\infty}^{\infty}g(y)e^{-\mathrm{i}\omega y}dy\right\} \\
= &\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\left\{\frac{1}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\left(\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)\widehat{g}(\omega)\right\}. \qquad (3.48)
\end{aligned}
$$

Similarly, we have

$$
\int_{-\infty}^{\infty} Q_g^{(4)}(y)e^{-\mathrm{i}\omega y}dy = \frac{\widetilde{d}_0}{r}\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}\widehat{g}(\omega). \qquad (3.49)
$$

By using integration by parts along with (3.47), we have for $\mu_2 \geq 1$

$$
\int_{-\infty}^{\infty} Q_g^{(2)}(y)e^{-\mathrm{i}\omega y}dy = -\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta b^2}{2r}\int_{-\infty}^{\infty}e^{-\mathrm{i}\omega x}g'(x)dx = -\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta \mathrm{i}\omega b^2}{2r}\widehat{g}(\omega), \qquad (3.50)
$$

and

$$
\int_{-\infty}^{\infty} Q_g^{(5)}(y)e^{-\mathrm{i}\omega y}dy = \frac{\widetilde{d}_0(-a)\mathrm{i}\omega}{r}\widehat{g}(\omega). \qquad (3.51)
$$

Furthermore, it is clear that for $\mu_2 \geq 1$

$$
\int_{-\infty}^{\infty} Q_g^{(3)}(y)e^{-\mathrm{i}\omega y}dy = -\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}(a+\frac{b^2\widetilde{\rho}_\eta}{2})\widehat{g}(\omega), \qquad (3.52)
$$

and

$$
\int_{-\infty}^{\infty} Q_g^{(6)}(y)e^{-\mathrm{i}\omega y}dy = \frac{\widetilde{d}_0(\lambda+\mu+r)}{r}\widehat{g}(\omega). \qquad (3.53)
$$

Combining (3.42), and (3.48)-(3.53), we see that

$$
\begin{aligned}
\widehat{Q}_g(\omega) &= \int_{-\infty}^{\infty} Q_g(y)e^{-\mathrm{i}\omega y}dy \\
&= \widehat{g}(\omega)\Bigg\{1_{\{\mu_2\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\Bigg[(\mathrm{i}\omega-\widetilde{\rho}_\eta)(\frac{-\mathrm{i}\omega b^2}{2}-a-\frac{b^2\widetilde{\rho}_\eta}{2}) \\
&\quad +\left(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\right)\Bigg] \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\left(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda\beta_k^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j}-\mathrm{i}a\omega+(\lambda+\mu+r)\right)\Bigg\}. \qquad (3.54)
\end{aligned}
$$

Using the facts $(\mathrm{i}\omega - \widetilde{\rho}_\eta)(\frac{-\mathrm{i}\omega b^2}{2} - a - \frac{b^2\widetilde{\rho}_\eta}{2}) = \frac{\omega^2 b^2}{2} - \mathrm{i}a\omega + a\widetilde{\rho}_\eta + \frac{b^2\widetilde{\rho}_\eta^2}{2}$ and (3.11), we obtain that

$$
\begin{aligned}
\widehat{Q}_g(\omega) &= \int_{-\infty}^{\infty} Q_g(y)e^{-\mathrm{i}\omega y}dy \\
&= \widehat{g}(\omega)\Bigg\{1_{\{\mu_2\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\Bigg[\frac{\omega^2 b^2}{2} - \mathrm{i}a\omega + a\widetilde{\rho}_\eta + \frac{b^2\widetilde{\rho}_\eta^2}{2} \\
&\quad +\Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j} + \frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\Bigg) + \Bigg(\sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\widetilde{\rho}_\eta)^m} - \frac{\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\mathrm{i}\omega)^m}\Bigg)\Bigg] \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\Bigg[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j} - \sum_{p=1}^{v_2}\sum_{m=1}^{\ell_p}\frac{\mu(\alpha_p)^m\widetilde{c}_{pm}}{(\alpha_p+\mathrm{i}\omega)^m} - \mathrm{i}a\omega + (\lambda+\mu+r)\Bigg]\Bigg\}.
\end{aligned}
$$

This together with (2.3) and (a) of Lemma 3.1 implies that

$$
\begin{aligned}
\widehat{Q}_g(\omega) &= \widehat{g}(\omega)\Bigg\{\frac{r-\psi_r(\omega)}{r}\Bigg(1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta} + 1_{\{a>0 \text{ and } b=0\}}\widetilde{d}_0\Bigg)\Bigg\} \\
&= \widehat{g}(\omega)\Bigg\{\Bigg(\frac{r-\psi_r(\omega)}{r}\Bigg)\psi_r^-(\omega)\Bigg\} = \widehat{g}(\omega)\Big(\psi_r^+(\omega)\Big)^{-1} \qquad (3.55)
\end{aligned}
$$

■

Remark 3.5 *(a) It follows from (3.54) and (3.55) that*

$$
\begin{aligned}
\Big(\psi_r^{(+)}(\omega)\Big)^{-1} =& 1_{\{\mu_2\geq 1\}}\frac{1}{r}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{\mathrm{i}\omega-\widetilde{\rho}_\eta}\Bigg[(\mathrm{i}\omega-\widetilde{\rho}_\eta)(\frac{-\mathrm{i}\omega b^2}{2} - a - \frac{b^2\widetilde{\rho}_\eta}{2}) + \Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j} + \frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\Bigg)\Bigg] \\
&+ 1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\Bigg[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\mathrm{i}\omega)^j} - \mathrm{i}a\omega + (\lambda+\mu+r)\Bigg].
\end{aligned}
$$

(b) It is interesting to notice that if $g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}$ *with* $0 \leq \max\{\theta_m : 1 \leq m \leq M\} < \rho_1$ *then* $Q_g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}\Big(\psi_r^+(-\mathrm{i}\theta_m)\Big)^{-1}$. *The result is consistent with [10], [5], and [15].*

In the following, we study some properties of $Q_g(x)$. We write $g \in \widehat{\pi}_0$ if $g \in \pi_0$ and g is nondecreasing and $g \in C^1(\widehat{a}, +\infty)$, where $\{g > 0\} = (\widehat{a}, +\infty)$ for some $\widehat{a} < \infty$.

Proposition 3.6 *Assume* $\{X_t\}_{t\geq 0}$ *is a jump-diffusion process of the form (2.2) with* $c_{kj} > 0$, $\beta_k > 0$, *and* $\alpha_p > 0$ *for* $1 \leq k \leq v_1, 1 \leq j \leq n_k, 1 \leq p \leq v_2$. *Consider the reward function* $g \in \widehat{\pi}_0$ *with* $\{g > 0\} = (\widehat{a}, \infty)$ *and assume* Q_g *is given by the formula in (3.21). Then*

(a) If there exists $\alpha > 0$ *such that* $\lim_{x\to\infty} Q_g(x) \geq \alpha$, *then there exists* $x^* > \widehat{a}$ *such that* $Q_g(x^*) = 0$.

(b) If $(\frac{g(u+x)}{g(x)})' < 0$ *and* $(\frac{g'(x)}{g(x)})' < 0$ *for any* $x > \widehat{a}$ *and* $u > 0$, *then there exists at most one* $x^* \in (\widehat{a}, +\infty)$ *such that* $Q_g(x^*) = 0$.

(c) If both conditions in (a) *and* (b) *hold, then there exists a unique* $x^* \in (\widehat{a}, +\infty)$ *such that* $Q_g(x^*) = 0$. *Moreover,* $Q_g(x)$ *is increasing for* $x > x^*$ *and* $Q_g(x) < 0$ *for* $\widehat{a} < x < x^*$.

Proof. Observe that

$$
\begin{aligned}
&Q_g(x)\\
=&\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{r(j-\ell)!}\left[\left(1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^\ell}+1_{\{a>0\text{ and }b=0\}}1_{\{\ell=1\}}\widetilde{d}_0\right)\int_0^\infty u^{j-\ell}g(u+x)e^{-\beta_k u}du\right]\\
&-\left[1_{\{\mu_2\geq 1\}}\frac{b^2}{2r}\left(\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta\right)+1_{\{a>0\text{ and }b=0\}}\frac{a\widetilde{d}_0}{r}\right]g'(x)\\
&-\left[1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}(a+\frac{b^2\widetilde{\rho}_\eta}{2})-1_{\{a>0\text{ and }b=0\}}\frac{\widetilde{d}_0}{r}(\lambda+\mu+r)\right]g(x). \qquad (3.56)
\end{aligned}
$$

We first show that $\lim_{t\to\widehat{a}^+}Q_g(t)<0$. To do this, we first claim that for $\mu_2\geq 1$ and $b\neq 0$

$$\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta=\frac{\prod_{j=1}^{\mu_2}-\widetilde{\rho}_j}{\prod_{k=1}^{v_2}(\alpha_k)^{\ell_k}}>0, \qquad (3.57)$$

and for $a>0$ and $b=0$

$$\widetilde{d}_0=\frac{\prod_{j=1}^{\mu_2}-\widetilde{\rho}_j}{\prod_{k=1}^{v_2}(\alpha_k)^{\ell_k}}>0. \qquad (3.58)$$

Also, we will show that for $1\leq k\leq v_1$ and $1\leq \ell\leq n_k$,

$$1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^\ell}+1_{\{a>0\text{ and }b=0\}}1_{\{\ell=1\}}\widetilde{d}_0>0. \qquad (3.59)$$

From identities (3.16) and (3.17), we acquire

$$\mathbb{E}[e^{uI_r}]=\prod_{k=1}^{v_2}\left(\frac{u+\alpha_k}{\alpha_k}\right)^{\ell_k}\prod_{j=1}^{\mu_2}\left(\frac{-\widetilde{\rho}_j}{u-\widetilde{\rho}_j}\right)=1_{\{\mu_2\geq 1\}}\sum_{j=1}^{\mu_2}\frac{\widetilde{d}_j\widetilde{\rho}_j}{u-\widetilde{\rho}_j}+1_{\{a>0\text{ and }b=0\}}\widetilde{d}_0. \qquad (3.60)$$

We obtain (3.57) by multiplying both sides by u, letting $u\to\infty$ in (3.60) and using the fact that $\mu_2=\sum_{k=1}^{v_2}\ell_k+1$. Similarly, (3.58) follows by letting $u\to\infty$ in (3.60) and using the fact that $\mu_2=\sum_{k=1}^{v_2}\ell_k$. To verify (3.59), we note that differentiating both sides of (3.60) at $u=\beta_k$ for ξ-times implies $\mathbb{E}[(I_r)^\xi e^{\beta_k I_r}]=1_{\{\mu_2\geq 1\}}(-1)^\xi\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^{\xi+1}}+1_{\{a>0\text{ and }b=0\}}1_{\{\xi=0\}}\widetilde{d}_0$. This yields (3.59). Using (3.57)-(3.59) and (3.56), we obtain $\lim_{t\to\widehat{a}^+}Q_g(t)<0$.

To prove (a), notice that by the assumption in (a), we have $\lim_{x\to+\infty}Q_g(x)>0$ and $Q_g(x)\in C(\widehat{a},+\infty)$. These together with $\lim_{t\to\widehat{a}^+}Q_g(t)<0$ and the intermediate-value theorem, imply that there exists at least one x^* in $(\widehat{a},\infty)$ such that $Q_g(x^*)=0$.

To prove (b), we write $Q_g(x)=g(x)h(x)$ for $x\in(\widehat{a},\infty)$, where

$$
\begin{aligned}
&h(x)\\
=&\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{r(j-\ell)!}\left[\left(1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{(\beta_k-\widetilde{\rho}_\eta)^\ell}+1_{\{a>0\text{ and }b=0\}}1_{\{\ell=1\}}\widetilde{d}_0\right)\int_0^\infty u^{j-\ell}\frac{g(u+x)}{g(x)}e^{-\beta_k u}du\right]\\
&-\left[1_{\{\mu_2\geq 1\}}\frac{b^2}{2r}\left(\sum_{\eta=1}^{\mu_2}\widetilde{d}_\eta\widetilde{\rho}_\eta\right)+1_{\{a>0\text{ and }b=0\}}\frac{a\widetilde{d}_0}{r}\right]\frac{g'(x)}{g(x)}\\
&-\left[1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}(a+\frac{b^2\widetilde{\rho}_\eta}{2})-1_{\{a>0\text{ and }b=0\}}\frac{\widetilde{d}_0}{r}(\lambda+\mu+r)\right].
\end{aligned}
$$

Taking account of the equations (3.57)-(3.59) and the conditions $(\frac{g(u+x)}{g(x)})' < 0$, $(\frac{g'(x)}{g(x)})' < 0$ for any $x > \widehat{a}$ and $u > 0$, we see that $h'(x) > 0$ for any $x > \widehat{a}$. This implies that there exists at most one $x^* \in (\widehat{a}, \infty)$ such that $h(x^*) = 0$. Hence, $Q_g(x) = 0$ has at most one solution in $(\widehat{a}, +\infty)$.

To prove (c), by (a) and (b) we see that there exists only one $x^* \in (\widehat{a}, \infty)$ such that $Q_g(x^*) = 0$. Furthermore, since $\lim_{t\to\widehat{a}^+} Q_g(t) < 0$, Q_g is continuous on $(\widehat{a}, \infty)$ and $Q_g(x) = 0$ has an unique solution on $(\widehat{a}, \infty)$, we have $Q_g(x) < 0$, for $x \in (\widehat{a}, x^*)$. For $x > x^*$, we have $Q_g(x) = g(x)h(x)$ and hence $Q_g'(x) = g'(x)h(x)+g(x)h'(x)$. Since each term of the right hand side is nonnegative, and $g(x)$ and $h'(x)$ are positive, we obtain $Q_g'(x) > 0$ for $x > x^*$ and hence $Q_g(x)$ is increasing on (x^*, ∞). ■

Combining Theorem 2.1, Theorem 3.3, and Proposition 3.6 gives the following main result.

Theorem 3.7 *Assume $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form (2.2) with $c_{kj} > 0$, $\beta_k > 0$, and $\alpha_p > 0$ for $1 \leq k \leq v_1, 1 \leq j \leq n_k, 1 \leq p \leq v_2$. Consider a reward function $g(x) \in \widehat{\pi}_0$ with $\{g > 0\} = (\widehat{a}, \infty)$ and $Q_g(x)$ given by (3.21). Assume that the following conditions hold:*

(a) There exists $\alpha > 0$ such that $\lim_{x\to\infty} Q_g(x) \geq \alpha$.

(b) $(\frac{g(u+x)}{g(x)})' < 0$ and $(\frac{g'(x)}{g(x)})' < 0$ for any $x > \widehat{a}$ and $u > 0$.

Then the optimal stopping time for the optimal stopping problem (1.1) is given by $\tau^* = \inf\{t > 0 : X_t > x^*\}$ and the value function is given by

$$V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})) = \int_{x^*-x}^{\infty} Q_g(x+m) f_{M_r}(m) dm.$$

Here x^* is the unique solution of the equation $Q_g(x) = 0$ in $(\widehat{a}, \infty)$ and f_{M_r} is given by (3.4).

4 Examples

In the following, we apply our results to some concrete examples. In particular we reproduce the special results of those discussed, among others, in Kyprianou and Surya[8], Mordecki [10], and Boyarchenko and Levendorskiĭ [5]. In all examples below, we always assume that $\{X_t\}_{t\geq 0}$ is a jump-diffusion process of the form (2.2) with $c_{kj} > 0$, $\beta_k > 0$, and $\alpha_p > 0$ for $1 \leq k \leq v_1, 1 \leq j \leq n_k$, and $1 \leq p \leq v_2$.

Example 4.1 *(Option with power function).*
Consider the optimal stopping problem (1.1) with $g(x) = (x^+)^\gamma$, $\gamma > 1$. According to (3.21), $Q_g(x)$ *is given by the formula*

$$\begin{aligned} Q_g(x) \;&=\; 1_{\{\mu_2\geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg[\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} \int_0^\infty u^{j-\ell} e^{-\beta_k u} (u+x)^\gamma du \\ &\qquad - \Big((a + \frac{b^2 \widetilde{\rho}_\eta}{2}) x^\gamma + \frac{b^2}{2} \gamma x^{\gamma-1} \Big) \Bigg] \\ &\quad + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} \int_0^\infty u^{j-1} e^{-\beta_k u} (u+x)^\gamma du \\ &\qquad + (\lambda + \mu + r) x^\gamma - a\gamma x^{\gamma-1} \Bigg\}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \lim_{x\to\infty} Q_g(x) x^{-\gamma} \;&=\; 1_{\{\mu_2\geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \Bigg\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell \beta_k^{j-\ell+1}} - (a + \frac{b^2 \widetilde{\rho}_\eta}{2}) \Bigg\} \\ &\quad + 1_{\{a>0 \text{ and } b=0\}} \frac{\widetilde{d}_0}{r} (\mu + r). \end{aligned}$$

By using the identity

$$\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\Bigg(\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-\theta)^{j-\ell+1}}\Bigg)=\frac{1}{\theta-\widetilde{\rho}_\eta}\Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\theta)^j}+\frac{\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^j}\Bigg) \tag{4.1}$$

and Remark 3.5(a), we see that $\lim_{x\to\infty}Q_g(x)x^{-\gamma}=\Bigg(\psi_r^+(0)\Bigg)^{-1}=1$*, which implies that* $\lim_{x\to\infty}Q_g(x)=\infty$*. Also, observe that for* $x>0$ *and* $u>0$*,* $\Bigg(\frac{g(u+x)}{g(x)}\Bigg)'=\Bigg((1+\frac{u}{x})^\gamma\Bigg)'=\gamma(1+\frac{u}{x})^{\gamma-1}(\frac{-u}{x^2})<0$ *and* $\Bigg(\frac{g'(x)}{g(x)}\Bigg)'=(\gamma x^{-1})'=-\gamma x^{-2}<0$*. By Theorem 3.7, there exists a unique* x^* *such that* $Q_g(x^*)=0$ *and* $\tau^*:=\inf\{t\geq 0: X_t\geq x^*\}$ *is the optimal stopping time for the optimal stopping problem (1.1) with* $g(x)=(x^+)^\gamma,\gamma>1$*.*

■

Remark 4.2 *Assume that* $g(x)=(x^+)^n$*, where* $n\in\mathbb{N}\cup\{0\}$*. Write* $Q_n(x)=Q_g(x)$*. Direct calculations show that* $Q_n(x)$ *satisfies* $Q_0(x)=\Bigg(\psi_r^+(0)\Bigg)^{-1}=1$*,* $\frac{d}{dx}Q_n(x)=nQ_{n-1}(x)$ *and* $\mathbb{E}\big[Q_n(M_r)\big]=0$*. Hence the functions* $Q_n(x)$ *are just the Appell polynomials for the random variable* M_r *in [8]. For Appell functions of any order* $\gamma\neq 0$ *and related works, see Novikov and Shiryaev [13] and Deligiannidis et al. [7].*

In the following example, we consider a special jump-diffusion model so that we can obtain a simple form for the value function.

Example 4.3 *Consider the case that* $g(x)=(x^+)^\gamma$ *with* $\gamma>1$*, and* $X_t=at+\sum_{i=1}^{N_t^\lambda}Y_i^\beta$ *where* $a<0$ *and* $\{Y_i^\beta: i=1,2,...\}$ *is a sequence of independent exponentially-distributed random variables with parameter* β*. Under these model assumptions, we have* $\psi(z)=\mathrm{i}az-\frac{\lambda z}{z+\mathrm{i}\beta}$ *and* $f_{M_r}(y)=d_0\delta_0(dy)+d_1\rho_1e^{-\rho_1 y}dy$*, where* $d_1=\frac{\beta-\rho_1}{\beta}$*,* $d_0=\frac{\rho_1}{\beta}$*, and* $\{-\mathrm{i}\rho_1,-\mathrm{i}\widetilde{\rho}_1\}$ *are the solutions of* $r-\psi(z)=0$*. Also, we have* $Q_g(x)=\frac{-\widetilde{\rho}_1}{r}\Bigg(-\frac{\lambda\beta}{\beta-\widetilde{\rho}_1}\int_x^\infty y^\gamma e^{-\beta(y-x)}dy-ax^\gamma\Bigg)$*, for every* $x>0$*. Hence, for each* $x<x^*$*, the value function is given by the formula*

$$\begin{aligned}V(x)&=\mathbb{E}\Bigg(Q_g(x+M_r)1_{\{x+M_r>x^*\}}\Bigg)=d_1\rho_1\int_{x^*}^\infty e^{-\rho_1(y-x)}Q_g(y)dy\\&=\frac{-(\beta-\rho_1)\rho_1\widetilde{\rho}_1}{\beta r}e^{\rho_1 x}\int_{x^*}^\infty e^{-\rho_1 y}\Bigg(-ay^\gamma-\frac{\lambda\beta}{\beta-\widetilde{\rho}_1}e^{\beta y}\int_y^\infty z^\gamma e^{-\beta z}dz\Bigg)dy.\end{aligned}$$

Since $\int_{x^*}^\infty e^{(\beta-\rho_1)y}\int_y^\infty z^\gamma e^{-\beta z}dzdy=\frac{1}{\beta-\rho_1}\Bigg(\int_{x^*}^\infty e^{-\rho_1 z}z^\gamma-e^{(\beta-\rho_1)x^*}\int_{x^*}^\infty e^{-\beta z}z^\gamma dz\Bigg)$*, we see that*

$$\begin{aligned}V(x)&=\frac{-(\beta-\rho_1)\rho_1\widetilde{\rho}_1}{\beta r}e^{\rho_1 x}\Bigg[(-a-\frac{\lambda\beta}{(\beta-\rho_1)(\beta-\widetilde{\rho}_1)})\int_{x^*}^\infty e^{-\rho_1 z}z^\gamma dz\\&\qquad+\frac{\lambda\beta}{(\beta-\rho_1)(\beta-\widetilde{\rho}_1)})e^{(\beta-\rho_1)x^*}\int_{x^*}^\infty e^{-\beta z}z^\gamma dz\Bigg].\end{aligned}$$

The fact that $-\mathrm{i}\rho_1$ *and* $-\mathrm{i}\widetilde{\rho}_1$ *are the solutions of* $r-\psi(z)=0$ *implies* $-a-\frac{\lambda\beta}{(\beta-\rho_1)(\beta-\widetilde{\rho}_1)}=0$*. This together with* $Q_g(x^*)=0$ *yields* $V(x)=\frac{a\widetilde{\rho}_1\rho_1}{\beta r}e^{\rho_1(x-x^*)}(x^*)^\gamma$*. Furthermore, since* $\psi(0)=0$ *and* $r-\psi(z)=\frac{(-\mathrm{i}a)(z+\mathrm{i}\rho_1)(z+\mathrm{i}\widetilde{\rho}_1)}{z+\mathrm{i}\beta}$*, we have* $\frac{a\widetilde{\rho}_1\rho_1}{\beta r}=1$*. Therefore, we obtain* $V(x)=e^{\rho_1(x-x^*)}(x^*)^\gamma$*. Clearly,*

V is continuous at the optimal boundary x^*. Since $V'(x^{*-}) = \rho_1(x^*)^\gamma$ and $g'(x^*) = \gamma(x^*)^{\gamma-1}$, there is no smooth fit at x^* as $x^* \neq \frac{\gamma}{\rho_1}$. (To show $x^* \neq \frac{\gamma}{\rho_1}$, we set $F(x) = \frac{-\lambda\beta}{a(\beta-\widetilde{\rho}_1)}\int_0^\infty(1+\frac{z}{x})^\gamma e^{-\beta z}dz$. Then using the inequality $(1+\frac{z}{x})^\gamma \leq e^{\frac{\gamma z}{x}}$ we observe that $F(x) < \frac{\lambda\beta}{(-a)(\beta-\widetilde{\rho}_1)} \cdot \frac{1}{\beta-\frac{\gamma}{x}}$. This implies that $F(\frac{\gamma}{\rho_1}) < \frac{\lambda\beta}{(-a)(\beta-\widetilde{\rho}_1)(\beta-\rho_1)} = 1$ and hence $Q_g(\frac{\gamma}{\rho_1}) > 0$. Consequently, $x^* < \frac{\gamma}{\rho_1}$.) Note that $\{0\}$ is not regular for the half-line $(0,\infty)$ for the process $\{X_t\}$. Our results show no contradiction with the general results of Theorem 5.1 in Surya [15].

■

Example 4.4 *(Perpetual American call option).*
We consider the optimal stopping problem (1.1) with $g(x) = (e^x - K)^+$ and assume that $\rho_1 > 1$. By (3.21), we have for $x > \ln K$

$$\begin{aligned}
Q_g(x) &= 1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\Bigg[\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(j-\ell)!}\int_0^\infty u^{j-\ell}e^{-\beta_k u}(e^{u+x}-K)du \\
&\qquad -\Big((a+\frac{b^2\widetilde{\rho}_\eta}{2})(e^x-K)+\frac{b^2}{2}e^x\Big)\Bigg] \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\Bigg\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!}\int_0^\infty u^{j-1}e^{-\beta_k u}(e^{u+x}-K)du \\
&\qquad +(\lambda+\mu+r)(e^x-K)-ae^x\Bigg\}.
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
\lim_{x\to\infty}Q_g(x)e^{-x} &= 1_{\{\mu_2\geq 1\}}\sum_{\eta=1}^{\mu_2}\frac{\widetilde{d}_\eta\widetilde{\rho}_\eta}{r}\Bigg\{\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\sum_{\ell=1}^{j}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-\widetilde{\rho}_\eta)^\ell(\beta_k-1)^{j-\ell+1}}-(a+\frac{b^2\widetilde{\rho}_\eta}{2})-\frac{b^2}{2}\Bigg\} \\
&\quad +1_{\{a>0 \text{ and } b=0\}}\frac{\widetilde{d}_0}{r}\Bigg(\sum_{k=1}^{v_1}\sum_{j=1}^{n_k}\frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k-1)^j}+(\lambda+\mu+r)-a\Bigg).
\end{aligned}$$

By using (4.1) and Remark 3.5(a), we see that $\lim_{x\to\infty}Q_g(x)e^{-x} = \Big(\psi_r^+(-\mathrm{i})\Big)^{-1}$. This along with $1 < \rho_1$ implies that $\lim_{x\to\infty}Q_g(x) = \infty$. Also, we have that for $x > \ln K$ and $u > 0$, $\Big(\frac{g(u+x)}{g(x)}\Big)' = \frac{Ke^x(e^x-e^{u+x})}{(e^x-K)^2} < 0$ and $\Big(\frac{g'(x)}{g(x)}\Big)' = \frac{-Ke^{2x}}{(e^x-K)^2} < 0$. Hence, by Theorem 3.7, we obtain the optimal stopping boundary x^* and the pricing formula in terms of Q_g and f_{M_r}. The solution was obtained earlier by Mordecki [10] for general Lévy processes.

■

Example 4.5 *We consider the optimal stopping problem (1.1) with $g(x) = \ln(x+1)1_{\{x\geq 0\}}$. To check conditions (a)-(b) in Theorem 3.7, we first substitute $g(x) = \ln(x+1)1_{\{x\geq 0\}}$ into (3.21). Multiplying both sides of (3.21) by $(\ln(x+1))^{-1}$ and using Remark 3.5(a), we see that $\lim_{x\to\infty}Q_g(x)(\ln(x+1))^{-1} = \Big(\psi_r^+(0)\Big)^{-1} = 1$, which implies that $\lim_{x\to\infty}Q_g(x) = \infty$. Next, observe that for $x > 0$ and $u > 0$*

$$\left(\frac{g(u+x)}{g(x)}\right)' = \left(\frac{\ln(u+x+1)}{\ln(x+1)}\right)' = \frac{\frac{\ln(x+1)}{u+x+1}-\frac{\ln(u+x+1)}{x+1}}{(\ln(x+1))^2} < 0,$$

and

$$\left(\frac{g'(x)}{g(x)}\right)' = \frac{-(x+1)^{-2}\cdot\ln(x+1)-(x+1)^{-2}}{(\ln(x+1))^{-2}} < 0.$$

By Theorem 3.7, there exists a unique $x^* > 0$ such that $Q_g(x^*) = 0$ and $\tau^* := \inf\{t \geq 0 : X_t \geq x^*\}$ is the optimal stopping time for the optimal stopping problem (1.1).

■

5 Appendix

Proof of (3.31).

$$
\begin{aligned}
& e^{\rho_s x} \int_x^z e^{(\beta_k - \rho_s)u} \int_u^\infty (y-u)^{j-\ell} g(y) e^{-\beta_k y} dy du \\
= & \sum_{\xi=1}^{j-\ell+1} \frac{(j-\ell)!}{(j-\ell+1-\xi)!(\beta_k-\rho_s)^\xi} \times \\
& \left[e^{\rho_s(x-z)} e^{\beta_k z} \int_z^\infty (y-z)^{j-\ell+1-\xi} g(y) e^{-\beta_k y} dy - e^{\beta_k x} \left(\int_x^\infty (y-x)^{j-\ell+1-\xi} g(y) e^{-\beta_k y} dy \right) \right] \\
& + \frac{(j-\ell)! e^{\rho_s x}}{(\beta_k - \rho_s)^{j-\ell+1}} \int_x^z g(y) e^{-\rho_s y} dy. \qquad (5.1)
\end{aligned}
$$

First, observe that $g \in \pi_0$ implies that the last term in (5.1) converges, as $z \to \infty$. So, it suffices to verify the first term converges to zero. Indeed, we have

$$
\begin{aligned}
\left| e^{\rho_s(x-z)} e^{\beta_k z} \int_z^\infty (y-z)^{j-\ell+1-\xi} g(y) e^{-\beta_k y} dy \right| &= \left| e^{\rho_s(x-z)} \int_0^\infty u^{j-\ell+1-\xi} g(u+z) e^{-\beta_k u} du \right| \\
&\leq \left| e^{\rho_s(x-z)} \right| \int_0^\infty u^{j-\ell+1-\xi} |g(u+z)| |e^{-\beta_k u}| du \\
&\leq \left| e^{\rho_s(x-z)} \right| \int_0^\infty u^{j-\ell+1-\xi} (A_1 + A_2 e^{\theta(u+z)}) |e^{-\beta_k u}| du.
\end{aligned}
$$

Taking account of the fact $0 < \theta < \rho_1 < \mathcal{R}e\,(\beta_k)$, we see that the last term above converges to zero, as $z \to \infty$. This yields $\lim_{z\to\infty} \left| e^{\rho_s(x-z)} e^{\beta_k z} \int_z^\infty (y-z)^{j-\ell+1-\xi} g(y) e^{-\beta_k y} dy \right| = 0$. The proof is complete. ■

References

[1] Alili, L. and Kyprianou, A.E. (2005). Some remarks on first passage of Lévy processes, the american put and pasting principles. *Ann. Appl. Prob.* **15,** 2062–2080.

[2] Asmussen, S. and Albrecher, H. (2010). *Ruin Probabilities.* 2nd edition. World Scientific Publishing Co. Pte. Ltd.

[3] Asmussen, S. (2003). *Applied Probability and Queues, Vol.***51** *of Applications of Mathematics* (New York). 2nd edition, Spinger-Verlag, Berlin.

[4] Asmussen, S., Avram, F. and Pistorius, M. (2004). Russian and American put options under exponential phase-type Lévy models. *Stoch. Process. Appl.* **109,** 79–111.

[5] Boyarchenko, S.I. and Levendorskiĭ, S.Z. (2002). Perpetual american options under Lévy processes. *SIAM J. Control Optim.* **40,** 1663–1696.

[6] Boyarchenko, S.I. and Levendorskiĭ, S.Z. (2005). American options: the EPV pricing model. *Ann. Finance.* **1,** 267–292.

[7] Deligiannidis, G., Le, H. and Utev, S. (2009). Optimal stopping for processes with independent increments, and applications. *J. Appl. Prob.* **46,** 1130–1145.

[8] Kyprianou, A.E. and Surya, B. (2005). On the Novikov-Shiryaev optimal stopping problems in continuous time. *Electron. Commun. Probab.* **10,** 146–154.

[9] Lewis, A. and Mordecki, E. (2008). Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms. *J. Appl. Prob.* **45,** 118–134.

[10] Mordecki, E. (2002). Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.* **6,** 473–493.

[11] Mordecki, E. (2003). Ruin probabilities for Lévy processes with mixed-exponential negative jumps. *Theory Prob. Appl.* **48,** 188–194.

[12] Novikov, A and Shiryaev, A.N. (2005). On an effecitive solution of the optimal stopping problem for random walks. *Theory Prob. Appl.* **49,** 344–354.

[13] Novikov, A and Shiryaev, A.N. (2007). On a solution of the optimal stopping problem for processes with independent increments. *Stochastics.* **79,** 393–406.

[14] Shiryayev, A.N. (1978). *Optimal Stopping Rules* (New York Inc. Springer-Verlag).

[15] Surya, B. A. (2007). An approach for solving perpetual optimal stopping problems driven by Lévy processes. *Stochastic.* **79,** 337–361.