THE CUTOFF PHENOMENON FOR EHRENFEST PROCESSES

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Abstract. We consider families of Ehrenfest chains and provide a simple criterion on the L^p-cutoff and the L^p-precutoff with specified initial states for $1 \le p < \infty$. For the family with an L^p-cutoff, a cutoff time is described and a possible window is given. For the family without an L^p-precutoff, the exact order of the L^p-mixing time is determined. The result is consistent with the well-known conjecture on cutoffs of Markov chains proposed by Peres in 2004, which says that a cutoff exists if and only if the multiplication of the spectral gap and the mixing time tends to infinity.

1. Introduction

Consider a time-homogeneous Markov chain on a finite set Ω with one-step transition matrix K. Let $K^t(x,\cdot)$ denote the probability distribution of the chain at time t starting from state x. It is well-known that if K is ergodic (irreducible and aperiodic), then

$$\lim_{t\to\infty} K^t(x,y)=\pi(y) \quad \forall x,y\in\Omega,$$

where π is the unique invariant probability of K on Ω. Denote by k_x^t the relative density of $K^t(x,\cdot)$ with respect to π, that is, $k_x^t(y)=K^t(x,y)/\pi(y)$. For $1\le p<\infty$, define the L^p-distance by

$$D_p(x,t)=\|k_x^t-1\|_{L^p(\pi)}=\left(\sum_{y\in\Omega}|k_x^t(y)-1|^p\pi(y)\right)^{1/p}.$$

For $p=\infty$, the L^∞-distance is set to be $D_\infty(x,t)=\max_y|k_x^t(y)-1|$. In the case $p=1$, this is exactly twice of the the total variation distance between $K^t(x,\cdot)$ and π, which is defined by

$$D_{\mathrm{TV}}(x,t)=\|K^t(x,\cdot)-\pi\|_{\mathrm{TV}}=\max_{A\subset\Omega}\{K^t(x,A)-\pi(A)\}.$$

For $p=2$, it is the so-called chi-square distance. For any $\epsilon>0$ and $1\le p\le\infty$, define the L^p-mixing time by

$$T_p(x,\epsilon)=\min\{t\ge 0: D_p(x,t)\le\epsilon\}.$$

The concept of cutoffs was introduced by Aldous and Diaconis in [1, 2, 3] to capture the fact that many ergodic Markov chains converge abruptly to their stationary distributions (in total variation and separation). We refer the reader to

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[6, 7, 13, 14, 15] for details and further discussions on variant examples. In a word, when $1 < p \leq \infty$, a family of finite ergodic Markov chains (Ω_n, K_n, π_n) with specified initial states x_n has an L^p-cutoff with cutoff time t_n if

$$\lim_{n\to\infty} D_{n,p}(x_n,(1+a)t_n) = \begin{cases} 0 & \text{if } a > 0 \\ \infty & \text{if } -1 < a < 0 \end{cases},$$

where $D_{n,p}$ denotes the L^p-distance of the nth Markov chain. The definition for cutoffs in total variation, separation and L^1-distance is the same as above expect the replacement of the limit ∞ with 1 in total variation and separation and with 2 in L^1-distance.

In [6], the authors discussed a number of variants of cutoffs and produced, in the reversible case, a necessary and sufficient condition for the existence of a max-L^p-cutoff, which is a cutoff in the distance $\max_{x\in\Omega} D_p(x,\cdot)$ with $1 < p \leq \infty$. In [7], there establishes an equivalent condition on the L^2-cutoff for families of Markov processes with specified initial distributions assuming the associated semigroups are normal. Also, a formula on the L^2-cutoff time was introduced in [7], based on a complete information of the spectral decomposition. This is in contrast with techniques and results in [6] which do not involve much in spectral theory.

Consider the Ehrenfest chains. For $n \geq 1$, let $\Omega_n = \{0, 1, ..., n\}$ and K_n be the Markov kernel of the Ehrenfest chain on Ω_n defined by

$$K_n(i,i+1) = 1 - \frac{i}{n}, \quad K_n(i+1,i) = \frac{i+1}{n}, \quad \forall 0 \leq i \leq n-1. \tag{1.1}$$

It is a simple exercise to check that the unbiased binomial distribution, $\pi_n(i) = \binom{n}{i}2^{-n}$, is the invariant probability of K_n and the pair (K_n, π_n) is reversible, i.e. $\pi_n(i)K_n(i,j) = \pi_n(j)K_n(j,i)$ for all $i, j \in \Omega_n$. By lifting the chain to a random walk on the hypercube, one may use the group representation of $(\mathbb{Z}_2)^n$ to identify the eigenvalues and eigenvectors of K_n as follows.

Lemma 1.1. *The matrix defined in* (1.1) *has eigenvalues*

$$\beta_{n,i} = 1 - \frac{2i}{n} \quad 0 \leq i \leq n,$$

with $L^2(\pi_n)$-normalized eigenvectors

$$\psi_{n,i}(x) = \binom{n}{i}^{-1/2} \sum_{k=0}^{i} (-1)^k \binom{x}{k}\binom{n-x}{i-k} \quad 0 \leq i, x \leq n. \tag{1.2}$$

See, e.g., [8] for a proof. Based on the above result, Chen and Saloff-Coste obtained the following theorem.

Theorem 1.2 ([7, Theorem 6.5])**.** *Let K_n be defined in* (1.1) *and set $K_n' = (I + nK_n)/(n+1)$, $\pi_n(i) = \binom{n}{i}2^{-n}$. Then, the following are equivalent.*

1. *The family $\{(\Omega_n, K_n', \pi_n) : n = 1, 2, ...\}$ with starting states $(x_n)_{n=1}^\infty$ has an L^2-cutoff.*
2. *$|n - 2x_n|/\sqrt{n} \to \infty$ as $n \to \infty$.*

Moreover, if (2) *holds, then*

$$T_{n,2}(x_n, \epsilon) = \frac{n}{2}\log\frac{|n-2x_n|}{\sqrt{n}} + O_\epsilon(n), \quad \forall \epsilon > 0.$$

The notation $O_\epsilon(n)$ denotes a sequence in n whose absolute values are bounded above by $C_\epsilon n$ for all $n \geq 1$ with $0 < C_\epsilon < \infty$.

The aim of this paper is to provide a necessary and sufficient condition on the L^p-cutoff of Ehrenfest chains with $1 \leq p < \infty$ and describe the L^p-cutoff time if any. For $1 < p < \infty$, the eigenfunctions are useful in bounding the L^p-distance but, however, they do not work very well in bounding the total variation distance of the associated semigroup from below. A path comparison to the simple random walk on $\mathbb{Z}$ is proposed to get suitable lower bound and this leads to the following result.

Theorem 1.3. *As in the setting of Theorem 1.2, the following are equivalent. For* $p \in [1, \infty)$,

(1) *The family* $\{(\Omega_n, K_n', \pi_n) : n = 1, 2, ...\}$ *with starting states* $(x_n)_{n=1}^\infty$ *has an* L^p*-cutoff.*
(2) *The family* $\{(\Omega_n, K_n', \pi_n) : n = 1, 2, ...\}$ *with starting states* $(x_n)_{n=1}^\infty$ *has an* L^p*-precutoff.*
(3) $|n - 2x_n|/\sqrt{n} \to \infty$ *as* $n \to \infty$.

Moreover, if (2) *holds, then*

$$T_{n,p}(x_n, \epsilon) = \frac{n}{2} \log \frac{|n - 2x_n|}{\sqrt{n}} + O_{\epsilon,p}(n), \quad \forall \epsilon > 0,\, p \in (1, \infty).$$

For $p = 1$*, the above identity remains true with* $\epsilon \in (0, 2)$.

This theorem is a special case of Theorem 4.1 and 5.1. The concept of precutoff will be introduced in the next section. In the case $p = 1$, it has been proved in [7] that (3) is sufficient for (1). As the Ehrenfest chain is a birth-and-death chain, we refer the reader to [9, 10] for more results on cutoffs, where the first article treats the cutoff in separation for chains starting from one end-point and the second article considers the max-total variation cutoff for lazy chains. Both of them introduce a universal criterion on cutoffs but the Ehrenfest chain is out of their categories.

The remaining of this article is organized in the following way. In Section 2, we recall various notions of cutoffs and quote useful results from [6]. In Section 3, we recall some well-known results for simple random walks on $\mathbb{Z}$, which will be used in latter context, and provide a proof on them. In Section 4, we deal with the total variation cutoff for the Ehrenfest chains in both the continuous time and discrete time cases. Those ideas inspired in this section are in fact applicable to more general models. In Section 5, we treat the L^p-cutoff and spell out the results along with the open problems.

2. Cutoffs

Throughout the oncoming sections, the term (Ω, K, π, μ) will be used to denote a time-homogeneous irreducible Markov chain on Ω with one-step transition matrix K, invariant probability π and initial distribution μ. Write (Ω, H_t, π, μ) as the continuous time Markov chain associated with (Ω, K, π, μ) if $H_t = e^{-t(I-K)}$, the semigroup associated with K. If the chain starts at state x, we write (Ω, K, π, x) and (Ω, H_t, π, x) instead. For any two sequences of positive numbers, say t_n, s_n, the notation $s_n = O(t_n)$ means that there are $N > 0$ and $C > 0$ such that $s_n \leq Ct_n$ for all $n \geq N$. If both $s_n = O(t_n)$ and $t_n = O(s_n)$ hold, we simply write $t_n \asymp s_n$. If $t_n/s_n \to 1$ as $n \to \infty$, write $t_n \sim s_n$ for short.

In this section, we recall various definitions of cutoffs and a series of related results from [6]. The notion of cutoff can be developed for any family of non-increasing functions taking values on $[0,\infty]$. The following definitions treat the L^p-cutoff for families of finite ergodic Markov chains with specified initial distributions in discrete time case. We refer the reader to [6] for further details and examples.

Definition 2.1. Let $\mathcal{F} = \{(\Omega_n, K_n, \pi_n, \mu_n) : n = 1, 2, ...\}$ be a family of irreducible and aperiodic finite Markov chains. For $p \in (1,\infty]$, the family $\mathcal{F}$ is said to present:

(1) An L^p-precutoff if there is a sequence $t_n > 0$ and constants $0 < A < B$ such that

$$\lim_{n\to\infty} D_{n,p}(\mu_n, B_n) = 0, \quad \liminf_{n\to\infty} D_{n,p}(\mu_n, A_n) > 0,$$

where $B_n = \inf\{j \ge 0 : j > Bt_n\}$ and $A_n = \sup\{j \ge 0 : j < At_n\}$.

(2) An L^p-cutoff if there is a sequence $t_n > 0$ such that, for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} D_{n,p}(\mu_n, \overline{k_n}(\epsilon)) = 0, \quad \lim_{n\to\infty} D_{n,p}(\mu_n, \underline{k_n}(-\epsilon)) = \infty,$$

where $\overline{k_n}(\epsilon) = \inf\{j \ge 0 : j > (1+\epsilon)k_n\}$ and $\underline{k_n}(\epsilon) = \sup\{j \ge 0 : j < (1+\epsilon)t_n\}$.

(3) A (t_n, b_n) L^p-cutoff if $t_n > 0$, $b_n > 0$, $b_n = o(t_n)$ and

$$\lim_{c\to\infty} \overline{F_p}(c) = 0, \quad \lim_{c\to-\infty} \underline{F_p}(c) = \infty,$$

where

$$\overline{F_p}(c) = \limsup_{n\to\infty} D_{n,p}(\mu_n, \overline{k}(n,c)), \quad \underline{F_p}(c) = \liminf_{n\to\infty} D_{n,p}(\mu_n, \underline{k}(n,c)),$$

and $\overline{k}(n,c) = \inf\{j \ge 0 : j > t_n + cb_n\}$ and $\underline{k}(n,c) = \sup\{j \ge 0 : j < t_n + cb_n\}$.

The definition for the case $p = 1$ follows if ∞ is replaced by 2.

The definition agrees with that in [6]. In (2) and (3), t_n is called an L^p-cutoff time and b_n is a window with respect to t_n. In (3), the functions, $\overline{F_p}$ and $\underline{F_p}$, give an idea on how the cutoff evolves and is sometimes called the shape of the (t_n, b_n) cutoff.

Remark 2.1. Note that, for $t > 0$, the mapping $t \mapsto D_{n,p}(\mu_n, t)$ is non-increasing. This implies that, if t_n tends to infinity (or equivalently $T_{n,p}(\mu_n, \epsilon) \to \infty$ for some $\epsilon > 0$) in Definition 2.1, it makes no difference to replace A_n with $\lfloor At_n \rfloor$ or $\lceil At_n \rceil$, and so for the replacements of B_n, $\overline{k_n}(\epsilon)$, $\underline{k_n}(\epsilon)$, $\overline{k}(n,c)$, and $\underline{k}(n,c)$.

Remark 2.2. In the continuous time case, the definition of cutoffs in Definition 2.1 follows in the intuitive way. That is, $A_n = At_n$, $B_n = Bt_n$, $\overline{k_n}(\epsilon) = \underline{k_n}(\epsilon) = (1+\epsilon)t_n$ and $\overline{k}(n,c) = \underline{k}(n,c) = t_n + cb_n$.

Remark 2.3. According to Definition 2.1, if a family has no L^p-precutoff (resp. L^p-cutoff), then the new family obtained by merging this one with any other still has no L^p-precutoff (resp. L^p-cutoff). This implies that if a subfamily has no L^p-precutoff (resp. L^p-cutoff), then the original family has no L^p-precutoff (resp. L^p-cutoff). But, however, there might exist another subfamily that has an L^p-precutoff (resp. L^p-cutoff).

Definition 2.2. Let (Ω, K, π, μ) be an irreducible finite Markov chain and $p \in [1,\infty]$. For $\epsilon > 0$, the ϵ-L^p-mixing time (or briefly the L^p-mixing time) is defined to be

$$T_p(\mu,\epsilon) := \inf\{t \geq 0 : D_p(\mu,t) \leq \epsilon\},$$

where the right side is set to be infinity if the infimum is taken over an empty set. If (Ω, H_t, π, μ) is the continuous time chain associated with K, write the L^p-mixing time as

$$T_p^c(\mu,\epsilon) := \inf\{t \geq 0 : D_p^c(\mu,t) \leq \epsilon\},$$

where $D_p^c(\mu,t)$ is the L^p-distance between μH_t and π.

The concept of cutoff can also be described using the notion of mixing time. For instance, assuming $T_{n,p}(\epsilon) \to \infty$ for some $\epsilon > 0$, a family of irreducible and aperiodic Markov chains has an L^p-cutoff if and only if

$$\lim_{n\to\infty} T_{n,p}(\mu_n,\epsilon)/T_{n,p}(\mu_n,\delta) = 1, \quad \forall \epsilon,\delta \in (0,M_p),$$

where $M_p = \infty$ if $p > 1$ and $M_1 = 2$. See [6, Proposition 2.3-2.4] for further details and relationships.

We end this section by introducing the following lemmas and corollary, which provide an idea on proving or disproving cutoffs.

Lemma 2.1 ([7, Proposition 2.1])**.** *Let $\mathcal{F} = \{(\Omega_n, K_n, \pi_n, \mu_n) : n = 1,2,...\}$ be a family of irreducible and aperiodic Markov chains. For any subsequence $\xi = (\xi_n)_{n=1}^{\infty}$ of positive integers, set $\mathcal{F}_\xi = \{(\Omega_{\xi_n}, K_{\xi_n}, \pi_{\xi_n}, \mu_{\xi_n}) : n = 1,2,...\}$. Let $p \in [1,\infty]$ and assume $T_{n,p}(\epsilon) \to \infty$ for some $\epsilon > 0$. Then, the following are equivalent.*

1. *$\mathcal{F}$ has an L^p-cutoff (resp. (t_n, b_n) L^p-cutoff).*
2. *For any subsequence ξ, $\mathcal{F}_\xi$ has an L^p-cutoff (resp. (t_{ξ_n}, b_{ξ_n}) L^p-cutoff).*
3. *For any subsequence ξ, there is a further subsequence ξ' such that $\mathcal{F}_{\xi'}$ has an L^p-cutoff (resp. $(t_{\xi'_n}, b_{\xi'_n})$ L^p-cutoff).*

Remark 2.4. In Lemma 2.1, (1)⇒(2)⇒(3) also holds true for the L^p-precutoff.

Lemma 2.2. *Let $\mathcal{F} = \{(\Omega_n, K_n, \pi_n, \mu_n) : n = 1,2,...\}$ be a family of irreducible and aperiodic Markov chains and $p \in [1,\infty]$. Suppose that there is $\epsilon > 0$ and $a_n \to \infty$ such that $T_{n,p}(\mu_n,\epsilon) \asymp a_n$ and $T_{n,p}(\mu_n,\delta) = O(a_n)$ for all $0 < \delta < \epsilon$. Then, the following are equivalent.*

1. *$\mathcal{F}$ has no L^p-precutoff.*
2. *For all $c > 0$,*
$$\limsup_{n\to\infty} D_{n,p}(\mu_n, \lfloor ca_n \rfloor) > 0.$$
3. *As $\delta \to 0$,*
$$\limsup_{n\to\infty} \frac{T_{n,p}(\mu_n,\delta)}{a_n} \to \infty.$$

Proof. (2)⇔(3) is obvious from the definition of the L^p-mixing time. By the monotonicity of the L^p-distance, the converse statements for (1) and (2) are exactly

(1)' $\mathcal{F}$ has an L^p-precutoff.

(2)' There is $C > 0$ such that

$$\lim_{n\to\infty} D_{n,p}(\mu_n, \lfloor Ca_n \rfloor) = 0.$$

We prove the equivalence of (1) and (2) by showing (1)'⇔(2)' instead. First, assume that $\mathcal{F}$ has an L^p-precutoff and, according to Remark 2.1, let $t_n > 0$ and $0 < A < B$ be constants such that

$$\liminf_{n\to\infty} D_{n,p}(\mu_n, \lfloor At_n \rfloor) = \epsilon_0 > 0, \quad \lim_{n\to\infty} D_{n,p}(\mu_n, \lfloor Bt_n \rfloor) = 0.$$

Let $\delta < \min\{\epsilon, \epsilon_0\}$ and choose $N > 0, C_1 > 0$ such that

$$D_{n,p}(\mu_n, \lfloor At_n \rfloor) > \delta > D_{n,p}(\mu_n, \lfloor Bt_n \rfloor), \quad T_{n,p}(\mu_n, \delta) \le C_1 a_n, \quad \forall n \ge N.$$

The former implies $At_n \le T_{n,p}(\mu_n, \delta) \le Bt_n$ and, then,

$$Bt_n \le \frac{BT_{n,p}(\mu_n, \delta)}{A} \le \frac{BC_1}{A} a_n.$$

This yields

$$\limsup_{n\to\infty} D_{n,p}(\mu_n, \lfloor BC_1 a_n/A \rfloor) \le \limsup_{n\to\infty} D_{n,p}(\mu_n, \lfloor Bt_n \rfloor) = 0.$$

Second, assume (2)' and choose $C_2 > 0$ such that $T_{n,p}(\mu_n, \epsilon) \ge C_2 a_n$ and $a_n \ge 2/C_2$. Then, for $n \ge 1$,

$$D_{n,p}(\mu_n, \lfloor C_2 a_n/2 \rfloor) \ge D_{n,p}(\mu_n, \lfloor C_2 a_n - 1 \rfloor) \ge D_{n,p}(\mu_n, T_{n,p}(\mu_n, \epsilon) - 1) > \epsilon > 0.$$

This proves the L^p-precutoff. □

The following is a simple corollary from Lemma 2.2, which surveys the L^p-precutoff in a more strict way.

Corollary 2.3. *As in the setting of Lemma 2.2, the following are equivalent.*

(1) *No subfamily of $\mathcal{F}$ has an L^p-precutoff.*
(2) *For all $c > 0$,*
$$\liminf_{n\to\infty} D_{n,p}(\mu_n, \lfloor ca_n \rfloor) > 0.$$
(3) *As $\delta \to 0$,*
$$\liminf_{n\to\infty} \frac{T_{n,p}(\mu_n, \delta)}{a_n} \to \infty.$$

Remark 2.5. It makes no difference to replace $\lfloor ca_n \rfloor$ with $\lceil ca_n \rceil$ in (2) of Lemma 2.2 and Corollary 2.3.

Remark 2.6. Lemma 2.1-2.2 and Corollary 2.3 can be generalized to any family of non-increasing functions defined on $\{0, 1, 2, ...\}$ or $[0, \infty)$. In particular, they hold for the continuous time Markov chains without the assumption $T_{n,p}(\mu_n, \epsilon) \to \infty$ and $a_n \to \infty$.

3. Simple random walks on $\mathbb{Z}$

This section is contributed to the establishment of some frequently used inequality related to the simple random walk on integers. A simple random walk is a discrete time Markov chain $(X_n)_{n=0}^{\infty}$ whose transition matrix is given by

$$K(i, i+1) = K(i, i-1) = 1/2, \quad \forall i \in \mathbb{Z}.$$

For $m \ge 1$, let T_m be the first passage time to the set $\{\pm m\}$, i.e.

$$T_m = \inf\{n \ge 0 : X_n = m \text{ or } X_n = -m\}. \tag{3.1}$$

For the continuous time case, let $N(t)$ be a Poisson process with parameter 1 and independent of X_n and set $Y_t = X_{N(t)}$. Clearly, Y_t is a realization of the semigroup

$H_t = e^{-t(I-K)}$ associated with K and the first passage time to $\{\pm m\}$ is denoted by

$$\widetilde{T}_m = \inf\{t \geq 0 : Y_t = m \text{ or } Y_t = -m\}. \tag{3.2}$$

Theorem 3.1. *Let $T_m, \widetilde{T}_m$ be the random times defined in* (3.1)-(3.2) *and $\mathbb{P}_0$ be the conditional probability given the initial state is* 0*. Then, for any $b > 1$ and $m \geq 5$,*

$$\min\{\mathbb{P}_0(T_m > bm^2), \mathbb{P}_0(\widetilde{T}_m > bm^2)\} \geq e^{-2b}.$$

Remark 3.1. This theorem says that, regardless of discrete time or continuous time cases, the simple random walk starting from the origin never reaches $\pm m$ before time m^2 with positive probability uniformly over m.

To prove this theorem, we introduce the following proposition.

Proposition 3.2. *Let K be the transition matrix of an irreducible birth-and-death chain on $\{0, 1, ...\}$. For $m \geq 1$, let τ_m and $\widetilde{\tau}_m$ be respectively the first passage times to state m associated with the discrete time and continuous time chains. Let $\lambda_1, ..., \lambda_m$ be the eigenvalues of the submatrix of $I - K$ indexed by $\{0, 1, ..., m-1\}$. Then, $\lambda_i \in (0, 2)$ for $1 \leq i \leq m$, $\lambda_i \neq \lambda_j$ for $i \neq j$, and*

$$\mathbb{P}_0(\tau_m > k) = \sum_{i=1}^{m} \left(\prod_{j:j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) (1 - \lambda_i)^k \tag{3.3}$$

and

$$\mathbb{P}_0(\widetilde{\tau}_m > t) = \sum_{i=1}^{m} \left(\prod_{j:j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) e^{-t\lambda_i}. \tag{3.4}$$

Remark 3.2. The right side of (3.4) is exactly $\mathbb{P}(\widetilde{T} > t)$, where $\widetilde{T}$ is a sum of m independent exponential random variables with parameters $\lambda_1, ..., \lambda_m$. Assuming $\lambda_i \in (0, 1)$ for all $1 \leq i \leq m$, the right side of (3.3) is equal to $\mathbb{P}(T > k)$, where T is a sum of independent geometric random variables with success probabilities $\lambda_1, ..., \lambda_m$.

Proof of Proposition 3.2. The proof for the continuous time case is available in [4], while the proof for the discrete time case follows in the same spirit. □

Back to the setting of the simple random walk. Observe that

$$\mathbb{P}_0(T_m > k) = \mathbb{P}_0(|X_i| < m, \forall i \leq k), \quad \mathbb{P}_0(\widetilde{T}_m > t) = \mathbb{P}_0(|X_s| < m, \forall s \leq t).$$

By the symmetry of the walk starting from 0, one may collapse states $\pm i$ to achieve

$$\mathbb{P}_0(T_m > k) = \mathbb{P}'_0(\tau_m > k), \quad \mathbb{P}_0(\widetilde{T}_m > t) = \mathbb{P}'_0(\widetilde{\tau}_m > t),$$

where $\mathbb{P}'_0$ is the probability for the birth-and-death chain on $\{0, 1, ...\}$ with initial state 0 and transition matrix K' given by

$$K'(0, 1) = 1, \quad K'(i, i-1) = K'(i, i+1) = 1/2, \quad \forall i \geq 1.$$

Here, τ_m and $\widetilde{\tau}_m$ are the first passage times to state m associated with the discrete time and continuous time chains driven by K'. Applying the method introduced in

[11, Section XIV.5], the eigenvalues and eigenvectors for the submatrix of $I - K'$ indexed by $0, 1, ..., m-1$ are

$$\lambda_i = 1 - \cos\frac{(2i-1)\pi}{2m}, \quad \phi_i(j) = \cos\frac{(2i-1)(j-1)\pi}{2m}, \quad \forall i, j \in \{1, ..., m\}.$$

We first treat the continuous time case. Let $S_1, ..., S_m$ be independent exponential random variables with parameters λ_i. As a consequence of Proposition 3.2, replacing t with bm^2 yields

$$\mathbb{P}_0(\widetilde{T}_m > bm^2) = \mathbb{P}(S_1 + \cdots + S_m > bm^2) \geq \mathbb{P}(S_1 > bm^2) = e^{-bm^2\lambda_1} \geq e^{-2b},$$

where the last inequality uses the fact $1 - \cos t \leq t^2/2$. For the discrete time case, the periodicity of K', which is of period 2, implies $\lambda_i > 1$ for some i. This prevents us from doing the same reasoning as the continuous time case. An idea to erase the periodicity of K' is to consider the lazy walk with transition matrix $\frac{1}{2}(I + K')$, since the eigenvalues of the submatrix of $I - \frac{1}{2}(I + K')$ indexed by $\{0, ..., m-1\}$ are contained in $(0, 1)$. To see the detail, let $(X'_n)_{n=0}^\infty$ be the birth-and-death chain with transition matrix K' and define $Z_n = X'_{2n}/2$. Obviously,

$$\mathbb{P}'_0(Z_{n+1} = 1 | Z_n = 0) = \mathbb{P}'_0(X'_{2n+2} = 2 | X'_{2n} = 0) = 1/2.$$

For $i > 0$,

$$\mathbb{P}'_0(Z_{n+1} = i + 1 | Z_n = i) = \mathbb{P}'_0(X'_{2n+2} = 2i + 2 | X'_{2n} = 2i) = 1/4$$

and

$$\mathbb{P}'_0(Z_{n+1} = i - 1 | Z_n = i) = \mathbb{P}'_0(X'_{2n+2} = 2i - 2 | X'_{2n} = 2i) = 1/4$$

and, for $i \geq 0$,

$$\mathbb{P}'_0(Z_{n+1} = i | Z_n = i) = \mathbb{P}'_0(X'_{2n+2} = 2i | X'_{2n} = 2i) = 1/2.$$

This implies that given $X'_0 = 0$, or equivalently $Z_0 = 0$, $(Z_n)_{n=0}^\infty$ is a Markov chain on $\{0, 1, ...\}$ with initial state 0 and transition matrix $\frac{1}{2}(I + K')$. Furthermore, by the periodicity of K', if m is even and positive, then

$$\mathbb{P}'_0(\tau_m > k) = \mathbb{P}'_0(X'_i < m, \forall i \leq k) = \mathbb{P}'_0(Z_i < m/2, \forall i \leq \lfloor k/2 \rfloor).$$

If m is odd and $m > 1$, then

$$\mathbb{P}'_0(\tau_m > k) = \mathbb{P}'_1(X'_i < m, \forall i \leq k-1) = \mathbb{P}'_0(Z_i < (m-1)/2, \forall i \leq \lfloor (k-1)/2 \rfloor),$$

where the last equality uses the fact that, given $X'_0 = 1$, the process $(X'_{2n} - 1)_{n=1}^\infty$ has the same distribution as $(Z_n)_{n=1}^\infty$ with $Z_0 = 0$. Let τ'_m be the first passage time to m of the chain $(Z_n)_{n=0}^\infty$. Putting all above together yields

$$\mathbb{P}_0(T_m > k) = \mathbb{P}'_0(\tau_m > k) \geq \mathbb{P}'_0(\tau'_{\lfloor m/2 \rfloor} > \lfloor k/2 \rfloor).$$

Note that the eigenvalues of the submatrix of $I - \frac{1}{2}(I + K')$ indexed by $0, 1, ..., \lfloor m/2 \rfloor - 1$ are $\lambda_i/2 \in (0, 1)$, $1 \leq i \leq \lfloor m/2 \rfloor$. By Proposition 3.2, if $S'_1, ..., S'_{\lfloor m/2 \rfloor}$ are independent geometric random variables with success probabilities $\lambda_1/2, ..., \lambda_{\lfloor m/2 \rfloor}/2$, then, for any positive integer k,

$$\mathbb{P}_0(T_m > k) \geq \mathbb{P}(S'_1 + \cdots + S'_{\lfloor m/2 \rfloor} > \lfloor k/2 \rfloor) \geq \left(\frac{1 + \cos(\pi/(2\lfloor m/2 \rfloor))}{2}\right)^{\lfloor k/2 \rfloor}.$$

Replacing k with $\lfloor bm^2 \rfloor$, $b > 1$ and $m > 1$ gives

$$\mathbb{P}_0(T_m > bm^2) \geq \left(\frac{1 + \cos(\pi/(2\lfloor m/2 \rfloor))}{2} \right)^{\lfloor k/2 \rfloor} \geq \left(\frac{1 + \cos(\pi/(m-1))}{2} \right)^{bm^2/2}$$
$$= \left(\cos \frac{\pi}{2(m-1)} \right)^{bm^2} \geq \left(1 - \frac{\pi^2}{8(m-1)^2} \right)^{bm^2} \geq e^{-2b},$$

where the last inequality uses the fact $\log(1-t) \geq -12t/11$ for $t < 1/12$ and asks $m \geq 5$.

4. The total variation cutoff of Ehrenfest chains

This section is dedicated to the total variation cutoff of Ehrenfest chains. First, recall the setting in (1.1). For $n \geq 1$, let $\Omega_n = \{0, 1, ..., n\}$ and K_n be the transition matrix of the Ehrenfest chain on Ω_n given by

$$K_n(i, i+1) = 1 - \frac{i}{n}, \quad K_n(i+1, i) = \frac{i+1}{n}, \quad \forall 0 \leq i \leq n-1. \tag{4.1}$$

It is easy to see that K_n is irreducible with stationary distribution $\pi_n(i) = \binom{n}{i} 2^{-n}$ for $0 \leq i \leq n$ and of period 2. Concerning the periodicity of K_n and the semigroup associated with K_n, consider

$$K_n' = \frac{1}{n+1} I + \frac{n}{n+1} K_n, \quad H_{n,t} = e^{-t(I - K_n)} = \sum_{i=0}^{\infty} \left(e^{-t} \frac{t^i}{i!} \right) K_n^i. \tag{4.2}$$

The total variation distance between $(K_n')^t$ (resp. $H_{n,t}$) and π_n with initial state x_n is defined by

$$D_{n,\mathrm{TV}}(x_n, t) := \max_{A \subset \Omega_n} |(K_n')^t(x_n, A) - \pi_n(A)|$$

and

$$D_{n,\mathrm{TV}}^c(x_n, t) := \max_{A \subset \Omega_n} |H_{n,,t}(x_n, A) - \pi_n(A)|.$$

The total variation variation mixing time is set to be

$$T_{n,\mathrm{TV}}(x_n, \epsilon) := \min\{t \geq 0 : D_{n,\mathrm{TV}}(x_n, t) \leq \epsilon\}$$

and

$$T_{n,\mathrm{TV}}^c(x_n, \epsilon) := \min\{t \geq 0 : D_{n,\mathrm{TV}}^c(x_n, t) \leq \epsilon\}.$$

For $p \in [1, \infty]$, let $D_{n,p}, D_{n,p}^c$ and $T_{n,p}, T_{n,p}^c$ be the L^p-distances and the L^p-mixing time in the discrete and continuous time cases.

Remark 4.1. The coupling, a classical probabilistic technique, was introduced by Aldous and Diaconis to control and further to identify the total variation distance. See [2] and the references therein for details.

According to the above setting, it is clear that the total variation distance is exactly half of the L^1-distance and has 1 as its maximum. In the same spirit, the total variation cutoff is consistent with the L^1-cutoff and, thus, the definition is the same as in Definition 2.1 except the replacement of ∞ by 1. The following theorem deals with the total variation cutoff of Ehrenfest chains.

Theorem 4.1. *For $n \geq 1$, let $x_n \in \Omega_n$. Consider the families $\mathcal{F} = \{(\Omega_n, K_n', \pi_n, x_n) : n = 1, 2, ...\}$ and $\mathcal{F}_c = \{(\Omega_n, H_{n,t}, \pi_n, x_n) : n = 1, 2, ...\}$. Then, the following are equivalent.*

(1) *$\mathcal{F}$ (resp, $\mathcal{F}_c$) has a total variation precutoff.*
(2) *$\mathcal{F}$ (resp, $\mathcal{F}_c$) has a total variation cutoff.*
(3) $|n-2x_n|/\sqrt{n}\to\infty$.

Furthermore, if (3) *holds, then both $\mathcal{F}$ and $\mathcal{F}_c$ have a (t_n,n) total variation cutoff with*

$$t_n=\frac{n}{2}\log\frac{|n-2x_n|}{\sqrt{n}}.$$

Remark 4.2. The window size n is optimal in the sense that, if $\mathcal{F}$ or $\mathcal{F}_c$ has a (t_n,b_n) total variation cutoff, then $n=O(b_n)$. See [6] for details on variants of window optimality.

Proof of Theorem 4.1. (3)⇒(2) and the (t_n,n) total variation cutoff under (3) has been proved in [7]. (2)⇒(1) follows from the definition. For (1)⇒(3), we assume (3) fails and prove $\mathcal{F}$ and $\mathcal{F}_c$ have no total variation precutoff. By Remark 2.3, it suffices to show that, if $|x_n-n/2|/\sqrt{n}$ is bounded, then no subfamily of $\mathcal{F}$ and $\mathcal{F}_c$ has a total variation precutoff. The proof consists of three steps.

Step1: Bounding the total variation from above. Note that the total variation distance is bounded above by the chi-square distance. That is,

$$2D_{n,\mathrm{TV}}(x,t)\le D_{n,2}(x,t),\quad 2D^c_{n,\mathrm{TV}}(x,t)\le D^c_{n,2}(x,t).$$

Using the reversibility of K_n and Lemma 1.1, the L^2-distance can be expressed as follows.

$$\begin{aligned}[D_{n,2}(x,t)]^2&=\sum_{i=1}^n|\psi_{n,i}(x)|^2\left(1-\frac{2i}{n+1}\right)^{2t}\\&\le 2\sum_{i=1}^{\lfloor n/2\rfloor}|\psi_{n,i}(x)|^2\left(1-\frac{2i}{n+1}\right)^{2t}+\left(1-\frac{2}{n+1}\right)^{2t}\\&\le 2\sum_{i=1}^{\lfloor n/2\rfloor}|\psi_{n,i}(x)|^2e^{-4it/(n+1)}+e^{-4t/(n+1)},\end{aligned}$$

where $\psi_{n,i}$ is the function defined in (1.2) and the first inequality applies the identity $\psi_{n,n-i}(x)=(-1)^x\psi_{n,i}(x)$ for all $x,i\in\{0,1,...,n\}$. It is worthwhile to note that the summation in the last line is also an upper bound for the continuous time case since

$$\begin{aligned}[D^c_{n,2}(x,t)]^2&=\sum_{i=1}^n|\psi_{n,i}(x)|^2e^{-4it/n}\\&\le 2\sum_{i=1}^{\lfloor n/2\rfloor}|\psi_{n,i}(x)|^2e^{-4it/(n+1)}+e^{-4nt/(n+1)}.\end{aligned}$$

Observe that $\psi_{n,i}(x)=\binom{n}{i}^{1/2}P_i(x,1/2,n)$, where $P_i(x,p,n)$ is the Krawtchouk polynomial, i.e.

$$P_i(x,p,n)={}_2F_1\left(\begin{array}{c}-i,-x\\-n\end{array}\middle|\frac{1}{p}\right).$$

See [12] for the definition. Using the following recurrence relation

$$(n-2x)P_i(x,1/2,n)=(n-i)P_{i+1}(x,1/2,n)+iP_{i-1}(x,1/2,n),$$

we may rewrite

$$\psi_{n,i+1}(x) = \frac{n-2x}{\sqrt{n}} A_{n,i}\psi_{n,i}(x) - B_{n,i}\psi_{n,i-1}(x), \tag{4.3}$$

where

$$A_{n,i} = \sqrt{\frac{n}{(i+1)(n-i)}}, \quad B_{n,i} = \sqrt{\frac{i(n-i+1)}{(i+1)(n-i)}}.$$

Obviously, for $n \ge 2$ and $1 \le i < n$, $A_{n,i} \le 1$ and $B_{n,i} \le 1$. By setting $r = 1 + \sup_n\{|n-2x_n|/\sqrt{n}\} < \infty$, we obtain

$$|\psi_{n,i+1}(x_n)| \le (r-1)|\psi_{n,i}(x_n)| + |\psi_{n,i-1}(x_n)|, \quad \forall 1 \le i < n.$$

Along with the boundary condition,

$$|\psi_{n,0}(x_n)| = 1, \quad |\psi_{n,1}(x_n)| = |n-2x_n|/\sqrt{n} \le (r-1),$$

the above inequality yields

$$|\psi_{n,i}(x_n)| \le r^i, \quad \forall 0 \le i \le n.$$

Putting this back to the computation of the L^2-distance derives, for any positive integer $N \ge \frac{1}{4}\log(2r^2)$,

$$\begin{aligned}
&\max\{D_{n,\mathrm{TV}}(x_n, N(n+1)), D^c_{n,\mathrm{TV}}(x_n, N(n+1))\} \\
\le&\frac{1}{2}\left(2\sum_{i=1}^{\lfloor n/2\rfloor} r^{2i}e^{-4iN} + e^{-4nN}\right)^{1/2} \le \left(\frac{1}{2}\sum_{i=1}^{\infty} r^{2i}e^{-4iN}\right)^{1/2} \\
\le&\left(\frac{r^2e^{-4N}}{2(1-r^2e^{-4N})}\right)^{1/2} \le re^{-2N},
\end{aligned} \tag{4.4}$$

where the last inequality uses the fact $e^t \ge 1+t$ for $t \ge 0$. Hence, for all $\epsilon \in (0,1)$ and $n \ge 2$,

$$\max\{T_{n,\mathrm{TV}}(x_n,\epsilon), T^c_{n,\mathrm{TV}}(x_n,\epsilon)\} \le \lceil \tfrac{1}{2}\log\tfrac{2r}{\epsilon}\rceil(n+1).$$

Step 2: Bounding the total variation from below: Discrete time case. In this step, we treat the discrete time case. Note that K'_n can be interpreted in the following way. First, flip a coin with probability $n/(n+1)$ landing on heads and evolve the chain according to K_n if a head appears. If the tail shows up, then the chain keeps in current state. Since the coin has a high preference on heads, the periodicity of K_n still plays an important role in the evolution of K'_n. This implies that the set partitioned by the period is a candidate of the testing set for the total variation. In the case of Ehrenfest chains, the set is either even integers or odd integers. From the viewpoint of the spectral theory, the period of any reversible finite Markov chain is either 1 or 2. Assuming the reversibility, a chain is periodic if and only if -1 is an eigenvalue of its transition matrix. Intuitively, the eigenvector associated with -1 should be able to provide a good idea on the construction of a testing set for the total variation. This is not clear for general chains, but it is quite obvious for Ehrenfest chain. According to Lemma 1.1, $\psi_{n,n}(x) = (-1)^x$ is an eigenvector of K_n associated with the eigenvalue -1 and the sets, $\{x \in \Omega_n : \psi_{n,n}(x) > 0\}$ and $\{x \in \Omega_n : \psi_{n,n}(x) < 0\}$, are exactly the event numbers and the odd numbers in Ω_n. Due to the above discussion, we set $A_n = \{i \in \Omega_n : i \text{ is even}\}$ and let $\mathbf{1}_{A_n}$ be the indicating function of A_n. Clearly, $2 \cdot \mathbf{1}_{A_n} - 1 = \psi_{n,n}$ and

$$\begin{aligned} D_{n,\mathrm{TV}}(x_n, t) &\geq |(K_n')^t(x_n, A_n) - \pi_n(A_n)| \\ &= \tfrac{1}{2}|[(K_n')^t(x_n, \cdot) - \pi_n](2 \cdot \mathbf{1}_{A_n} - 1)| \\ &= \tfrac{1}{2}|(K_n')^t(x_n, \psi_{n,n})| \\ &= \tfrac{1}{2}(1 - \tfrac{2}{n+1})^t \geq \tfrac{1}{2}e^{-4t/(n+1)}, \end{aligned} \tag{4.5}$$

for $n \geq 3$, where the last inequality applies the fact $\log(1-t) \geq -2t$ for $t \in [0, 1/2]$. This implies, for $0 < \epsilon \leq 1/(2e^4)$,

$$T_{n,\mathrm{TV}}(x_n, \epsilon) \geq \lfloor \tfrac{1}{4} \log \tfrac{1}{2\epsilon} \rfloor (n+1), \quad \forall n \geq 3.$$

It is worthwhile to note that the lower bound is independent of the initial state.

Along with the upper bound in Step 1, we obtain $T_{n,\mathrm{TV}}(x_n, 1/(2e^4)) \asymp n$ and $T_{n,\mathrm{TV}}(x_n, \epsilon) = O_\epsilon(n)$ for all $\epsilon < 1/(2e^4)$. Using the last inequality of (4.5), it is easy to see that, for any $c \geq 1$ and $n \geq 1$,

$$D_{n,\mathrm{TV}}(x_n, \lfloor cn \rfloor) \geq D_{n,\mathrm{TV}}(x_n, \lfloor 2c \rfloor (n+1)) \geq \tfrac{1}{2}e^{-4\lfloor 2c \rfloor} \geq e^{-9c}.$$

By Corollary 2.3, no subfamily of $\mathcal{F}$ has a total variation precutoff.

Step 3: Bounding the total variation from below: Continuous time case. Again, we suppose $|n - 2x_n|/\sqrt{n}$ is bounded. It has been developed in Step 1 that $T^c_{n,\mathrm{TV}}(x_n, \epsilon) = O_\epsilon(n)$ for all $\epsilon \in (0, 1)$. By Corollary 2.3, it suffices to show that

$$\liminf_{n \to \infty} D^c_{n,\mathrm{TV}}(x_n, cn) > 0, \quad \forall c > 0. \tag{4.6}$$

The trick used in Step 2 does not work for the continuous time case, since, by writing

$$\exp\{-t(I - K_n)\} = \exp\left\{-2t\left[I - \left(\frac{I + K_n}{2}\right)\right]\right\},$$

the continuous time Markov chain behaves like the lazy chain, a Markov chain whose transition matrix has entries in the diagonal at least 1/2. Comparing with K_n', $(I + K_n)/2$ evolves according to a fair coin and K_n. That is, if the coin lands on heads, then the chain transits states according to K_n. If the coin lands on tails, then the chain keeps at current state. For lazy chains, their eigenvalues must be nonnegative and the smallest eigenvalue has less contribution to the L^2-distance and the total variation. Our policy to conquer the continuous time case is as follows. First, we compare the original discrete time Ehrenfest chain K_n with the simple random walk on $\mathbb{Z}$. Based on the symmetry of the Ehrenfest chain, the comparison will generate a lower bound on the total variation distance related to the first passage time discussion in Section 3. This will lead to (4.6).

First, observe that, for any $A \subset \Omega_n$ and $t \geq 0$,

$$D^c_{n,\mathrm{TV}}(x_n, t) \geq H_{n,t}(x_n, A) - \pi_n(A) = \sum_{i=0}^{\infty} \left(e^{-t}\frac{t^i}{i!}\right) K_n^i(x_n, A) - \pi_n(A). \tag{4.7}$$

By the symmetry of K_n and the boundedness of $|x_n - n/2|/\sqrt{n}$, it loses no generality to assume that $n/4 \leq x_n \leq n/2$ for all $n \geq 0$. Moreover, by Remark 2.4, it suffices to deal with the following subcases.

$$(n/2 - x_n)/\sqrt{n} \to a \in [0, \infty), \quad \text{as } n \to \infty. \tag{4.8}$$

The next proposition is helpful in the selection of the testing set A.

Proposition 4.2. *Let K_n be the transition matrix on Ω_n defined by* (4.1)*. Suppose μ_n is a probability concentrated on $A = \{0, 1, ..., \lceil n/2 \rceil\}$, i.e., $\mu_n(A) = 1$. Then, $\mu_n K_n^t(A) \geq 1/2$ for all $t \geq 0$.*

This proposition realizes the intuition that, by the symmetry of Ehrenfest chains, if the initial distribution concentrates on the left half side of Ω_n, then so does the distribution of the chain at all time. See the appendix for a proof of this proposition. Now, let $A = \{0, 1, ..., \lceil n/2 \rceil\}$. Clearly, $\pi_n(A) \leq 1/2 + \pi_n(\lceil n/2 \rceil)$ and, by Stirling's formula, $\pi_n(\lceil n/2 \rceil) \sim (\pi n/2)^{-1/2}$. Let T be the first passage time to state $\lfloor n/2 \rfloor$, the first time (including time 0) to hit $\lfloor n/2 \rfloor$, for the Ehrenfest chain K_n. The irreducibility of K_n implies $\mathbb{P}_{x_n}(T < \infty) = 1$ and the strong Markov property yields

$$K_n^i(x_n, A) = \sum_{j=0}^{i} K_n^{i-j}(\lfloor n/2 \rfloor, A)\mathbb{P}_{x_n}(T = j) + \mathbb{P}_{x_n}(T > i) \geq \frac{1}{2} + \frac{1}{2}\mathbb{P}_{x_n}(T > i).$$

Putting this back to (4.7), we obtain, for all $m \geq 0$,

$$D_{\mathrm{TV}}^c(x_n, t) \geq \frac{1}{2}\left(e^{-t}\sum_{i=0}^{m}\frac{t^i}{i!}\right)\mathbb{P}_{x_n}(T > m) - \pi_n(\lceil n/2 \rceil). \tag{4.9}$$

Next, we use Theorem 3.1 to bound $\mathbb{P}_{x_n}(T > m)$ from below. Consider the simple random walk on $\mathbb{Z}$. For $m \geq 1$, $k \geq 1$ and $i \in \mathbb{Z}$, let $\mathcal{P}(m, k, i)$ be the set containing paths of length m starting from 0, ending at i and staying in $\{0, \pm 1, \pm 2, ..., \pm(k-1)\}$ up to time m. Clearly,

$$\mathbb{P}_{x_n}(T > m) \geq \sum_{i=0}^{\lfloor n/2 \rfloor - x_n - 1} \mathbb{P}_{x_n}(\mathcal{P}(m, \lfloor n/2 \rfloor - x_n, i))$$

Let $\mathbb{P}'$ be the probability where the simple random walk on $\mathbb{Z}$ starting from the origin sits. For any path $w = (w_0, w_1, ..., w_m) \in \mathcal{P}(m, k, i)$ with $|i| < k$, one may partition the edges $\{(w_j, w_{j+1}) : 0 \leq k < m\}$ into two subsets, say $B_1(w)$ and $B_2(w)$, where $B_1(w) = \{(j, j+1) : 0 \leq j < i\}$ for $i > 0$, $B_1(w) = \{(j, j-1) : 0 \geq j > i\}$ for $i < 0$, and $B_2(w)$ is a disjoint union of pairs in the form $\{(j, j+1), (j+1, j)\}$ with $-k < j < k-1$. Note that, for $2x_n - n/2 \leq j \leq n/2$,

$$1 - \frac{j}{n} \geq \frac{j}{n} \geq \frac{1}{2}\left(\frac{4x_n}{n} - 1\right) = \frac{1}{2}\left(1 - \frac{2(n-2x_n)}{n}\right)$$

and

$$\left(1 - \frac{j}{n}\right)\frac{j+1}{n} \geq \frac{1}{4}\left[1 - \left(\frac{n-2j}{n}\right)^2\right] \geq \frac{1}{4}\left[1 - 4\left(\frac{n-2x_n}{n}\right)^2\right].$$

This leads to $\mathbb{P}_{x_n}(w) \geq c_n(m)\mathbb{P}'(w)$ for all $w \in \mathcal{P}(m, \lfloor n/2 \rfloor - x_n, i)$ and $2x_n - n/2 \leq i \leq n/2$, where

$$c_n(m) = \left[1 - 4\left(\frac{n-2x_n}{n}\right)^2\right]^m \left(1 - \frac{2(n-2x_n)}{n}\right)^{n/2 - x_n}.$$

Let $m = Nn$, where N is any positive integer. Using the notation in (3.1) and applying Theorem 3.1, we obtain

$$\begin{aligned}\mathbb{P}_{x_n}(T > Nn) &\geq c_n(Nn)\mathbb{P}'_0(T_{\lfloor n/2\rfloor - x_n} > Nn)\\ &\geq c_n(Nn)\exp\left\{-\frac{2Nn}{(\lfloor n/2\rfloor - x_n)^2}\right\},\end{aligned}$$

provided $Nn \geq (\lfloor n/2\rfloor - x_n)^2$. Putting this back to (4.9), we obtain

$$D^c_{\mathrm{TV}}(x_n,t) \geq \frac{1}{2}\left(e^{-t}\sum_{i=0}^{Nn}\frac{t^i}{i!}\right)c_n(Nn)\exp\left\{-\frac{2Nn}{(\lfloor n/2\rfloor - x_n)^2}\right\} - \pi_n(\lfloor n/2\rfloor),$$

if $Nn \geq (\lfloor n/2\rfloor - x_n)^2$. As a consequence of Lemma A.3, if $a > 0$ in the setting of (4.8), then

$$\liminf_{n\to\infty} D^c_{\mathrm{TV}}(x_n, cn) \geq \frac{1}{2}e^{-(20a^2+2/a^2)N} > 0, \quad \forall N > \max\{c, a^2, 1\}.$$

By Corollary 2.3, this prove that if $a > 0$, then no subfamily of $\mathcal{F}_c$ has a total variation precutoff.

In the end, we deal with the subcase $a = 0$. Obviously, the last inequality provides a trivial lower bound on the total variation. To get an applicable bound, we rewrite the transition density of K^t_n as follows using Lemma 1.1.

$$K^t_n(x,y)/\pi_n(y) - 1 = \sum_{i=1}^{n}\psi_{n,i}(x)\psi_{n,i}(y)|\beta_{n,i}|^t.$$

See [14, Lemma 1.3.3] for a proof. Applying this identity to the case $(K'_n)^t$ and $H_{n,t}$ gives

$$\frac{(K'_n)^t(x,y)}{\pi_n(y)} - 1 = \sum_{i=1}^{n}\psi_{n,i}(x)\psi_{n,i}(y)\left(\frac{1+n\beta_{n,i}}{n+1}\right)^t \tag{4.10}$$

and

$$\frac{H_{n,t}(x,y)}{\pi_n(y)} - 1 = \sum_{i=1}^{n}\psi_{n,i}(x)\psi_{n,i}(y)e^{-t(1-\beta_{n,i})} \tag{4.11}$$

For $n \geq 1$, set

$$H_{n,t}(x_n,y)/\pi_n(y) - 1 = f_n(t,y) + g_n(t,y),$$

where

$$f_n(t,y) = \psi_{n,2}(x_n)e^{-t(1-\beta_{n,2})}\psi_{n,2}(y)$$

and

$$g_n(t,y) = \sum_{i=1, i\neq 2}^{n}\psi_{n,i}(x_n)e^{-t(1-\beta_{n,i})}\psi_{n,i}(y).$$

As a consequence of the triangle inequality and Jensen's inequality, we obtain

$$2D^c_{\mathrm{TV}}(x_n,t) = \|f_n(t,\cdot) + g_n(t,\cdot)\|_{L^1(\pi_n)} \geq \|f_n(t,\cdot)\|_{L^1(\pi_n)} - \|g_n(t,\cdot)\|_{L^2(\pi_n)}.$$

It remains to prove that, for all $c > 0$,

$$\liminf_{n\to\infty}\left[\|f_n(cn,\cdot)\|_{L^1(\pi_n)} - \|g_n(cn,\cdot)\|_{L^2(\pi_n)}\right] > 0.$$

First, observe that

$$\|g_n(t,\cdot)\|_{L^2(\pi_n)} = \left(\frac{n-2x_n}{\sqrt{n}}e^{-4t/n} + \sum_{i=3}^{n} |\psi_{n,i}(x_n)|^2 e^{-4it/n}\right)^{1/2}.$$

Recall the following fact developed in Step 1. If $r = 1 + \sup_n\{|n-2x_n|/\sqrt{n}\} < \infty$, then

$$|\psi_{n,i}(x_n)| \le r^i, \quad \forall 0 \le i \le n.$$

Putting this back to the $L^2(\pi_n)$-norm of $g_n(t,\cdot)$ yields

$$\|g_n(cn,\cdot)\|_{L^2(\pi_n)} \le \left(\frac{n-2x_n}{\sqrt{n}}e^{-4c} + \frac{(re^{-4c})^3}{1-re^{-4c}}\right)^{1/2},$$

provided $r < e^{4c}$. Also, it is an easy exercise to compute

$$\psi_{n,2}(x) = \sqrt{\frac{n}{2(n-1)}}\left[\left(\frac{n-2x}{\sqrt{n}}\right)^2 - 1\right].$$

This implies $|\psi_{n,2}(x_n)| \sim 1/\sqrt{2}$ and

$$\|\psi_{n,2}\|_{L^1(\pi_n)} \ge \frac{1}{2}\pi_n(\{x : |x-n/2| < \sqrt{n}/4\}) \sim \frac{1}{\sqrt{2\pi}}\int_0^{1/2} e^{-u^2/2}du \ge \frac{1}{12}.$$

According to the assumption $(n/2 - x_n)/\sqrt{n} \to a = 0$, if $r < e^{4c}$, then

$$\begin{aligned}&\liminf_{n\to\infty}\left[\|f_n(cn,\cdot)\|_{L^1(\pi_n)} - \|g_n(cn,\cdot)\|_{L^2(\pi_n)}\right] \\ \ge &\frac{1}{12\sqrt{2}}e^{-4c} - \frac{r^{3/2}}{\sqrt{1-re^{-4c}}}e^{-6c} = e^{-4c}\left(\frac{1}{12\sqrt{2}} - \frac{r^{3/2}}{\sqrt{1-re^{-4c}}}e^{-2c}\right) > 0,\end{aligned}$$

for c large enough. By the monotonicity of the total variation distance, we have

$$\liminf_{n\to\infty} D^c_{\mathrm{TV}}(x_n, cn) > 0, \quad \forall c > 0.$$

By Corollary 2.3, no subfamily of $\mathcal{F}_c$ has a total variation precutoff when $a = 0$. This finishes the proof. □

Remark 4.3. In the proof of Theorem 4.1, it has been shown that if $|x_n - n/2|/\sqrt{n}$ is bounded, then no subfamily of $\mathcal{F}$ and $\mathcal{F}_c$ presents a total variation precutoff and the total variation mixing time is of order n.

Remark 4.4. In Step 3, the method for $a = 0$ is also valid for $a > 0$ if one replaces $f_n(t,\cdot)$ with $\psi_{n,1}(x_n)e^{-t(1\beta_{n,1})}\psi_{n,1}$ and changes $g_n(t,\cdot)$ into $H_{n,t}(x_n,\cdot)/\pi_n - 1 - f_n$. The proof for $a > 0$ also works for the discrete time case.

5. The L^p-cutoff of Ehrenfest chains

This section is contributed to the development of the L^p-cutoff of Ehrenfest chains with $p \in (1,\infty)$. To bound the L^p-distance, we have to select suitable test functions in accordance with the operator theory and the spectral information provides some good ideas on the choice, for instance, the eigenfunctions. The main theorem states as follows.

Theorem 5.1. *Let $\mathcal{F}$ and $\mathcal{F}_c$ be the families in Theorem 4.1. For $p \in (1,\infty)$, the following are equivalent.*

(1) *$\mathcal{F}$ (resp. $\mathcal{F}_c$) has an L^p-precutoff.*

(2) *$\mathcal{F}$ (resp. $\mathcal{F}_c$) has an L^p-cutoff.*
(3) $|x_n - n/2|/\sqrt{n} \to \infty$.

Moreover, if (3) *holds, then both $\mathcal{F}$ and $\mathcal{F}_c$ have a (t_n, n) L^p-cutoff with*

$$t_n = \frac{n}{2}\log\frac{|n-2x_n|}{\sqrt{n}}.$$

Proof. In this proof, we will write $\|f\|_p$ as the $L^p(\pi)$-norm of f for short. Obviously, (2)⇒(1) comes immediate from Definition 2.1 for all $1 < p < \infty$. For (3)⇒(2) and the (t_n, n) L^p-cutoff, we set

$$\overline{F_p}(a) = \limsup_{n\to\infty} D_{n,p}(x_n, t_n + an), \quad \underline{F_p}(a) = \liminf_{n\to\infty} D_{n,p}(x_n, t_n + an)$$

and

$$\overline{G_p}(a) = \limsup_{n\to\infty} D^c_{n,p}(x_n, t_n + an), \quad \underline{G_p}(a) = \liminf_{n\to\infty} D^c_{n,p}(x_n, t_n + an).$$

Consider in the following two cases, $p \in (1, 2]$ and $p \in (2, \infty)$.

Case 1: $(1 < p \le 2)$ For $p = 2$, (2) and (3) have been proved equivalent in [7]. In detail, by Theorem 6.3-6.5 in [7] and the proofs therein, there are positive constants A, N such that, for $n \ge N$,

$$\max\{D_{n,2}(x_n, t_n + an), D^c_{n,2}(x_n, t_n + an)\} \le Ae^{-2a} + o(1)$$

and

$$\min\{D_{n,2}(x_n, t_n + an), D^c_{n,2}(x_n, t_n + an)\} \ge e^{-2a} + o(1).$$

This implies

$$\max\{\overline{F_2}(a), \overline{G_2}(a)\} \le Ae^{-2a}, \quad \min\{\underline{F_2}(a), \underline{G_2}(a)\} \ge e^{-2a}, \quad \forall a \in \mathbb{R}. \tag{5.1}$$

Note that the L^r-distance is bounded above by L^s-distance for $1 \le r < s \le \infty$. Using the first inequality of (5.1), we obtain, for $p \in (1, 2)$,

$$\max\{\overline{F_p}(a), \overline{G_p}(a)\} \le Ae^{-2a} \to 0, \quad \text{as } a \to \infty.$$

To get a lower bound, consider the test function $\psi_{n,1}$. Set $q = (1 - 1/p)^{-1}$. A simple application of the central limit theorem yields

$$\|\psi_{n,1}\|_q = \left(\sum_{x=0}^{n}\left(\frac{|n-2x|}{\sqrt{n}}\right)^q \pi_n(x)\right)^{1/q} \to C_q := [\mathbb{E}(|X|^q)]^{1/q},$$

where X is a standard normal random variable and $\mathbb{E}$ denotes the expectation. It is a simple exercise to show that

$$C_q = \left(\sqrt{\frac{2^q}{\pi}}\Gamma\left(\frac{q+1}{2}\right)\right)^{1/q} < \infty, \quad \forall q \in (1, \infty),$$

where Γ is the Gamma function defined by $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$. As a consequence of (4.10)-(4.11), we have

$$\underline{F_p}(a) \ge \liminf_{n\to\infty} \frac{|\langle (K_n')^{t_n+an}(x_n, \cdot)/\pi_n - 1, \psi_{n,1}\rangle_{\pi_n}|}{\|\psi_{n,1}\|_q} = e^{-2a}/C_q$$

and

$$\underline{G_p}(a) \ge \liminf_{n\to\infty} \frac{|\langle H_{n,t_n+an}(x_n, \cdot)/\pi_n - 1, \psi_{n,1}\rangle_{\pi_n}|}{\|\psi_{n,1}\|_q} = e^{-2a}/C_q.$$

Obviously, $\min\{\underline{F_p}(a), \underline{G_p}(a)\} \to \infty$ as $a \to -\infty$. This proves the desired (t_n, n) L^p-cutoff.

Case 2: $(2 < p < \infty)$ Using the second inequality of (5.1), it is easy to see that

$$\min\{\underline{F_p}(a), \underline{G_p}(a)\} \ge e^{-2a} + o(1) \to \infty, \quad \text{as } a \to -\infty.$$

To get an upper bound, we apply the fact $\psi_{n,n-i}(x) = (-1)^x \psi_{n,i}(x)$ to the right sides of (4.10)-(4.11) and get

$$D_{n,p}(x_n, t) \le 2 \sum_{i=1}^{\lceil n/2 \rceil} |\psi_{n,i}(x_n)| \|\psi_{n,i}\|_p \left(1 - \frac{2i}{n+1}\right)^t + \left(1 - \frac{2}{n+1}\right)^t \le 2d_p(n,t)$$

and

$$D^c_{n,p}(x_n, t) \le 2 \sum_{i=1}^{\lceil n/2 \rceil} |\psi_{n,i}(x_n)| \|\psi_{n,i}\|_p e^{-2it/n} + \left(1 - \frac{2}{n+1}\right)^t \le 2d_p(n,t),$$

where

$$d_p(n,t) = \sum_{i=1}^{\lceil n/2 \rceil} |\psi_{n,i}(x_n)| \|\psi_{n,i}\|_p e^{-2it/(n+1)} + e^{-2t/(n+1)}.$$

To bound $d_p(n,t)$ from above, one has to compute the L^p-norm of $\psi_{n,i}$. This can be very complicated from its definition but, surprisingly, the identity in (4.3) is sufficient to give a reasonable upper bound. In detail, one may derive from (4.3) that, for $i \le n/2$,

$$|\psi_{n,i+1}(x)| \le \left(\sqrt{\frac{2}{i+1}} \times \frac{|n-2x|}{\sqrt{n}}\right) |\psi_{n,i}(x)| + |\psi_{n,i-1}(x)|.$$

Along with the initial conditions, $\psi_{n,0} \equiv 1$ and $\psi_{n,1}(x) = (n-2x)/\sqrt{n}$, an inductive argument yields

$$|\psi_{n,i}(x)| \le \sqrt{\frac{2^i}{i!}} \prod_{j=1}^{i} \left(|\psi_{n,1}(x)| + \sqrt{\frac{j}{2}}\right), \quad \forall x \in \Omega_n,\ i \le n/2. \tag{5.2}$$

For convenience, write $i! = \alpha_i i^{i+1/2} e^{-i}$. By Stirling's formula, $\alpha_i \to \sqrt{2\pi}$ as $i \to \infty$. Thus, we may choose $\beta > 1$ such that $\beta^{-1} \le \alpha_i \le \beta$ for all $i \ge 1$. This implies

$$i^{i+1/2} e^{-i}/\beta \le i! \le \beta i^{i+1/2} e^{-i}, \quad \forall i \ge 1. \tag{5.3}$$

In this setting, (5.2) gives

$$|\psi_{n,i}(x)| \le (2e)^{i/2} i^{-1/4} \beta^{1/2} \left(|\psi_{n,1}(x)| i^{-1/2} + 1\right)^i \tag{5.4}$$

and, then, the L^p-norm of $\psi_{n,i}$ is bounded above as follows.

$$\begin{aligned} \|\psi_{n,i}\|_p^p &\le (2e)^{pi/2} i^{-p/4} \beta^{p/2} \pi_n \left[\left(|\psi_{n,1}| i^{-1/2} + 1\right)^{pi}\right] \\ &\le (2e)^{pi/2} i^{-p/4} \beta^{p/2} 2^{pi} \left[i^{-pi/2} \pi_n \left(|\psi_{n,1}|^{pi}\right) + 1\right], \end{aligned}$$

where the last inequality uses the fact $(s+t)^r \le 2^{r-1}(s^r + t^r)$ for any $s > 0$, $t > 0$ and $r \ge 1$. It deserves to note that, for fixed i, the central limit theorem implies that $\pi_n(|\psi_{n,1}|^{pi})$ converges to the expectation of $|X|^{pi}$, where X is the standard normal random variable. To estimate such a convergence for all $1 \le i \le n$, one may consider the convergence rate of the central limit theorem, but, however, this can be very complicated. Here, we cook up a direct computation in Lemma A.4, which says that there exists a constant $C > 1$ such that

$$\pi_n(|\psi_{n,1}|^{pi}) \le C4^{pi}\Gamma\left(\frac{pi+1}{2}\right).$$

As a consequence of the identity $\Gamma(t+1) = t\Gamma(t)$,

$$\begin{aligned}\Gamma\left(\frac{pi+1}{2}\right) &\le 2\prod_{j=1}^{\lfloor (pi-1)/2\rfloor}\frac{pi-2j+1}{2}\\ &\le pi\times\left(\left\lceil\frac{pi-3}{2}\right\rceil!\right) \le 5\beta pi[(pi)/(2e)]^{pi/2}.\end{aligned}$$

For $p \ge 2$, the above inequalities gives

$$\|\psi_{n,i}\|_p \le \left((2e)^{pi/2}i^{-p/4}\beta^{p/2}2^{pi}\{20\beta C4^{pi}(pi)[p/(2e)]^{pi/2}\}\right)^{1/p} \le 10\beta Ci^{1/4}(8p)^i.$$

Plugging the last term and (5.4) back to $d_p(n,t)$, we obtain

$$d_p(n,t) \le 10\beta^2 C\sum_{i=1}^{\lceil n/2\rceil}(20p)^i\left(|\psi_{n,1}(x_n)|+1\right)^i e^{-2it/(n+1)} + e^{-2t/(n+1)}. \tag{5.5}$$

Recall that

$$t_n = \frac{n}{2}\log\frac{|n-2x_n|}{\sqrt{n}} = \frac{n}{2}\log|\psi_{n,1}(x_n)|.$$

Clearly, for $a > 1$,

$$t_n + an \ge \frac{n+1}{2}\log|\psi_{n,1}(x_n)| + (a-1)n \ge \frac{n+1}{2}\log|\psi_{n,1}(x_n)| + \frac{n+1}{2}(a-1).$$

This implies

$$d_p(n,t_n+an) \le 10\beta^2 C\sum_{i=1}^{\lceil n/2\rceil}\left(\frac{20p}{e^{a-1}}\times\frac{|\psi_{n,1}(x_n)|+1}{|\psi_{n,1}(x_n)|}\right)^i + \exp\{-|\psi_{n,1}(x_n)|\}.$$

Under the assumption of (3), that is, $|\psi_{n,1}(x_n)| \to \infty$, if $e^{a-1} > 8p$, then

$$\begin{aligned}\max\{\overline{F_p}(a),\overline{G_p}(a)\} &\le 2\limsup_{n\to\infty} d_p(n,t_n+an)\\ &\le 20\beta^2 C\sum_{i=1}^{\infty} i(20pe^{1-a})^i = \frac{400\beta^2 Cpe^{1-a}}{1-20pe^{1-a}}.\end{aligned}$$

Obviously, the last term converges to 0 as a tends to infinity. This proves the (t_n, n) L^p-cutoff of $\mathcal{F}$ and $\mathcal{F}_c$ with $2 < p < \infty$.

For (1)⇒(3), we assume that $|x_n - n/2|/\sqrt{n}$ is bounded and prove that no subfamily of $\mathcal{F}$ and $\mathcal{F}_c$ has an L^p-precutoff. Set $M = \sup_{n\ge1}\{|2x_n - n|/\sqrt{n}\} + 1$. By (5.5), we have, for $p > 2$ and $e^a \ge 20Mp$

$$\begin{aligned}&\max\{D_{n,p}(x_n,\lceil an\rceil), D^c_{n,p}(x_n,an)\} \le 2d_p(n,an)\\ \le&20\beta^2 C\sum_{i=1}^{\infty}(20Mpe^{-a})^i + 2e^{-a} = \frac{400M\beta^2 Cpe^{-a}}{1-20Mpe^{-a}} + 2e^{-a}.\end{aligned}$$

Again, the right side converges to 0 as a tends to infinity. This implies, for all $\epsilon > 0$ and $p < \infty$,

$$T_{n,p}(x_n,\epsilon) = O_\epsilon(n), \quad T^c_{n,p}(x_n,\epsilon) = O_\epsilon(n).$$

Also, by Remark 4.3 and Corollary 2.3, we have

$$\liminf_{n\to\infty} \min\{D_{n,\mathrm{TV}}(x_n, cn), D^c_{n,\mathrm{TV}}(x_n, cn)\} > 0, \quad \forall c > 0.$$

This yields, for $p > 1$,

$$\liminf_{n\to\infty} \min\{D_{n,p}(x_n, cn), D^c_{n,p}(x_n, cn)\} > 0, \quad \forall c > 0.$$

Consequently, for $1 < p < \infty$, no subfamily of $\mathcal{F}$ and $\mathcal{F}_c$ has an L^p-precutoff. This finishes the proof. □

Remark 5.1. It is worthwhile to note that if $|n - x_n|/\sqrt{n}$ is bounded, then the L^p-mixing time of the Ehrenfest chains in (4.2) with $p \in [1, \infty)$ is of order n.

Remark 5.2. For $p = \infty$, the equivalence in Theorem 5.1 might fall. Suppose n is even, $x_n = n/2$ and consider the separation distance, which is closely related to the L^∞-distance and is defined by

$$D_{n,\mathrm{sep}}(x, t) = \max_y \left\{1 - \frac{(K'_n)^t(x, y)}{\pi_n(y)}\right\}, \quad D^c_{n,\mathrm{sep}}(x, t) = \max_y \left\{1 - \frac{H_{n,t}(x, y)}{\pi_n(y)}\right\}.$$

For $n \ge 1$, let L_n be a Markov kernel on $\{0, 1, ..., n/2\}$ given by

$$L_n(i, i) = 0, \quad \forall 0 \le i \le n/2, \quad L_n(i, i + 1) = 1 - \frac{i}{n}, \quad \forall 0 \le i < n/2,$$

and

$$L_n(i + 1, i) = \frac{i + 1}{n}, \quad \forall 0 \le i < n/2 - 1, \quad L_n(n/2, n/2 - 1) = 1.$$

It is obviously that L_n is obtained from K_n by collapsing states $\{i, n - i\}$ and has $\widetilde{\pi}_n(i) = 2^{1-n}\binom{n}{i}$ for $i < n/2$ and $\widetilde{\pi}_n(n/2) = 2^{-n}\binom{n}{n/2}$ as the stationary distribution. Let $\widetilde{D}_{n,\mathrm{sep}}(x, t), \widetilde{D}^c_{n,\mathrm{sep}}(x, t)$ be respectively the separation distances between $(L'_n)^t, e^{-t(I-L_n)}$ and $\widetilde{\pi}_n$, where $L'_n = (I + nL_n)/(n + 1)$. Then,

$$D_{n,\mathrm{sep}}(n/2, t) = \widetilde{D}_{n,\mathrm{sep}}(n/2, t), \quad D^c_{n,\mathrm{sep}}(n/2, t) = \widetilde{D}^c_{n,\mathrm{sep}}(n/2, t).$$

In fact, the above identities also hold in the L^p-distance with $1 \le p \le \infty$. In [9], the authors consider discrete time monotone birth-and-death chains, which is not satisfied by L'_n, and continuous time birth-and-death chains without any constraint. It is an easy exercise to check that $I - L_n$ has eigenvalues $4i/n$ and eigenvectors $\phi_{n,i}$ given by $\phi_{n,i}(x) = \psi_{n,2i}(x)$ for $0 \le i \le n/2$. Clearly, the spectral gap of L_n is $\lambda_n = 4/n$. Set

$$t_n = \sum_{i=1}^{n/2} \frac{n}{4i} = \frac{n \log n}{4} + O(n).$$

As a consequence of [9, Theorem 5.1-6.1], the family $\mathcal{F}_c$ in Theorem 4.1 has a $(\frac{1}{4}n \log n, n)$ separation cutoff. However, according to Theorem 5.1 and Remark 5.1, $\mathcal{F}_c$ has no L^p-precutoff and the exact order of the L^p-mixing time is n.

Remark 5.3. There is no universal criterion on the total variation cutoff or precutoff, except specific chains such as lazy birth-and-death chains. Concerning the maximum total variation distance and the related mixing time, define

$$D_{\mathrm{TV}}(t) = \max_{x\in\Omega} D_{\mathrm{TV}}(x, t), \quad T_{\mathrm{TV}}(\epsilon) = \inf\{t \ge 0 : D_{\mathrm{TV}}(t) \le \epsilon\}$$

and call the cutoff in the above distance as the maximum total variation cutoff. The authors of [10] prove that a family of lazy birth-and-death chains on $\Omega_n = \{0, 1, ..., n\}$ has a maximum total variation cutoff if and only if

$$\lim_{n\to\infty} \lambda_n T_{n,\mathrm{TV}}(\epsilon) = \infty,$$

for some $\epsilon \in (0, 1)$, where $1 - \lambda_n$ is the second largest eigenvalue of the transition matrix on Ω_n. Such a criterion is proposed by Peres during the ARCC workshop held by AIM in Palo Alto, December 2004. Under the assumption of reversibility, it has been shown to be true in [6] for max-L^p distance with $1 < p < \infty$, but disproved in [5] for $p = 1$ using an idea from Aldous. However, none of the above results is clear if the initial states or distributions for a family of ergodic Markov chains are specified. As a consequence of Theorem 4.1, Lemma 1.1 and Remark 4.3, the family in Theorem 4.1 has a total variation cutoff (also for the precutoff) if and only if

$$\lim_{n\to\infty} \lambda_n T_{n,\mathrm{TV}}(x_n, \epsilon) = \infty,$$

for some $\epsilon \in (0, 1)$. This provides an example that is consistent with Peres' conjecture.

Appendix A. Techniques and proofs

We consider Proposition 4.2 in a more general setting.

Lemma A.1. *Let K be the transition matrix of a periodic birth-and-death chain on $\Omega = \{0, 1, ..., m\}$ with birth rate p_i and death rate $q_i = 1 - p_i$. That is,*

$$K(i, i+1) = p_i, \quad K(i, i-1) = q_i = 1 - p_i, \quad \forall 0 \le i \le m,$$

with the convention $p_m = q_0 = 0$. Let $l = \lfloor m/2 \rfloor$ and μ be a probability on Ω. Suppose that, for any $i \ge 0$,

$$\mu(l-2i) \ge \mu(l+2i+2) \ge \mu(l-2i-2), \quad p_{l+2i} \ge q_{l-2i} \ge p_{l+2i+2}, \tag{A.1}$$

and

$$p_{l+2i} + q_{l+2i+2} \ge p_{l-2i-2} + q_{l-2i} \ge p_{l+2i+2} + q_{l+2i+4}. \tag{A.2}$$

Then, for all $i \ge 0$,

$$\mu K(l+2i+1) \ge \mu K(l-2i-1) \ge \mu K(l+2i+3).$$

Proof. By the periodicity of K,

$$\mu K(j) = \mu(j-1)p_{j-1} + \mu(j+1)q_{j+1}, \quad \forall 0 \le j \le m,$$

where

$$\mu(-1) = \mu(m+1) = p_{-1} = q_{m+1} = 0. \tag{A.3}$$

It is easy to check that both (A.1) and (A.2) hold under the extension in (A.3). If $i \le (l-1)/2$, then $l + 2i + 1 \le 2l \le m$ and

$$\begin{aligned}
&\mu K(l+2i+1) - \mu K(l-2i-1) \\
=&[\mu(l+2i)p_{l+2i} + \mu(l+2i+2)q_{l+2i+2}] \\
&\quad - [\mu(l-2i)q_{l-2i} + \mu(l-2i-2)p_{l-2i-2}] \\
\ge&\mu(l-2i)(p_{l+2i} - q_{l-2i}) + \mu(l+2i+2)(q_{l+2i+2} - p_{l-2i-2}) \\
\ge&\mu(l+2i+2)(p_{l+2i} - q_{l-2i} + q_{l+2i+2} - p_{l-2i-2}) \ge 0.
\end{aligned}$$

If $l+2i+3 \le m$, then $l-2i-1 \ge 2l+2-m \ge 1$ and

$$\begin{aligned}&\mu K(l-2i-1)-\mu K(l+2i+3)\\=&[\mu(l-2i)q_{l-2i}+\mu(l-2i-2)p_{i-2i-2}]\\&\quad-[\mu(l+2i+2)p_{l+2i+2}+\mu(l+2i+4)q_{l+2i+4}]\\\ge&\mu(l+2i+2)(q_{l-2i}-p_{l+2i+2})+\mu(l-2i-2)(p_{i-2i-2}-q_{l+2i+4})\\\ge&\mu(l-2i-2)(q_{l-2i}-p_{l+2i+2}+p_{i-2i-2}-q_{l+2i+4})\ge 0.\end{aligned}$$

This finishes the proof. □

Remark A.1. Lemma A.1 also holds for the case that m is even and $l = m/2-1$. The proof goes similarly and is omitted.

The following is a simple corollary of Lemma A.1.

Corollary A.2. *Let K be the transition matrix on $\Omega = \{0,1,...,m\}$ given by*

$$K(i,i+1)=p_i,\quad K(i,i-1)=q_i=1-p_i,\quad \forall 0\le i\le m,$$

where $p_m = q_0 = 0$, and let μ be a probability on Ω. Suppose that

$$p_i = q_{m-i},\quad p_i \ge p_{i+1},\quad \forall i \ge 0,$$

and

$$p_i+q_{i+2}\le p_{i+1}+q_{i+3},\quad \forall 0\le i\le \lfloor m/2\rfloor-2.$$

(1) *If $m = 2l$ and*
$$\mu(l+2i)\ge\mu(l-2i-2)\ge\mu(l+2i+2),\quad\forall i\ge 0,$$
then, for all $i \ge 0$ and $t \in \{0,1,2,...\}$,
$$\mu K^{2t+1}(l-2i-1)\ge\mu K^{2t+1}(l+2i+1)\ge\mu K^{2t+1}(l-2i-3)$$
and
$$\mu K^{2t}(l+2i)\ge\mu K^{2t}(l-2i-2)\ge\mu K^{2t}(l+2i+2).$$

(2) *If $m = 2l$ and*
$$\mu(l-2i-1)\ge\mu(l-2i+1)\ge\mu(l-2i-3),\quad\forall i\ge 0,$$
then, for all $i \ge 0$ and $t \in \{0,1,2,...\}$,
$$\mu K^{2t+1}(l+2i)\ge\mu K^{2t+1}(l-2i-2)\ge\mu K^{2t+1}(l+2i+2).$$
and
$$\mu K^{2t}(l-2i-1)\ge\mu K^{2t}(l+2i+1)\ge\mu K^{2t}(l-2i-3).$$

(3) *If $m = 2l+1$ and*
$$\mu(l-2i)\ge\mu(l+2i+2)\ge\mu(l-2i-2),\quad\forall i\ge 0,$$
then, for all $i \ge 0$ and $t \in \{0,1,2,...\}$,
$$\mu K^{2t+1}(l+2i+1)\ge\mu K^{2t+1}(l-2i-1)\ge\mu K^{2t+1}(l+2i+3)$$
and
$$\mu K^{2t}(l-2i)\ge\mu K^{2t}(l+2i+2)\ge\mu K^{2t}(l-2i-2).$$

Proof of Proposition 4.2. For the birth-and-death chain in Proposition 4.2, it is obvious that $p_i = 1 - i/n$ and $q_i = i/n$. This implies

$$p_i = q_{n-i}, \quad p_i > p_{i+1}, \quad p_i + q_{i+2} = 1 + \frac{2}{n}, \quad \forall i \geq 0.$$

Applying Corollary A.2 with $K = K_n$ and $\mu = \delta_{\lceil n/2 \rceil}$, the dirac mass on $\lceil n/2 \rceil$, yields

$$K_n^t(\lceil n/2 \rceil, A) \geq 1/2, \quad \forall t \geq 0.$$

For the general case with $\mu_n(A) \geq 1/2$, let $(X_t)_{t=0}^{\infty}$ be a Markov chain with transition matrix K_n and let T be the first passage time to state $\lceil n/2 \rceil$, i.e., $T = \min\{t \geq 0 : X_t = \lceil n/2 \rceil\}$. By the irreducibility of K_n, $\mathbb{P}_{\mu_n}(T < \infty) = 1$. Using the strong Markov property, we obtain, for $t \geq 0$,

$$\begin{aligned}
\mu_n K_n^t(A) =& \sum_{i=0}^{t} \mathbb{P}_{\mu_n}(X_t \in A, T = i) + \mathbb{P}_{\mu_n}(X_t \in A, T > t) \\
=& \sum_{i=0}^{t} \mathbb{P}(X_{t-i} \in A | X_0 = \lceil n/2 \rceil) \mathbb{P}_{\mu_n}(T = i) + \mathbb{P}_{\mu_n}(T > t) \\
\geq& \tfrac{1}{2} \mathbb{P}_{\mu_n}(T \leq t) + \mathbb{P}_{\mu_n}(T > t) \geq 1/2.
\end{aligned}$$

□

Lemma A.3 ([6, Lemma A.1])**.** *For $n > 0$, let $a_n \in \mathbb{R}^+$, $b_n \in \mathbb{Z}^+$, $c_n = \frac{b_n - a_n}{\sqrt{a_n}}$ and $d_n = e^{-a_n} \sum_{i=0}^{b_n} \frac{a_n^i}{i!}$. Assume that $a_n + b_n \to \infty$. Then*

$$\limsup_{n \to \infty} d_n = \Phi\left(\limsup_{n \to \infty} c_n\right), \quad \liminf_{n \to \infty} d_n = \Phi\left(\liminf_{n \to \infty} c_n\right), \tag{A.4}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$.

In particular, if c_n converges(the limit can be $+\infty$ and $-\infty$), then $\lim_{n \to \infty} d_n = \Phi\left(\lim_{n \to \infty} c_n\right)$.

Lemma A.4. *For $n \geq 1$, let ξ_n be a binomial random variable with parameters $(n, 1/2)$. Then, there is a universal constant $C > 0$ such that*

$$E\left(\left|\frac{n - 2\xi_n}{\sqrt{n}}\right|^\theta\right) \leq C 4^\theta \Gamma\left(\frac{\theta + 1}{2}\right), \quad \forall \theta > 0,\, n \geq 1,$$

where Γ is the Gamma function.

Proof. Set $\Omega_n = \{0, 1, ..., n\}$ and $\pi_n(x) = \binom{n}{x} 2^{-n}$. According to the definition of ξ_n, $\mathbb{P}(\xi_n = x) = \pi_n(x)$ for $x \in \Omega_n$. For $0 \leq j < \sqrt{n}$, set

$$E_{n,j} = \{x \in \Omega_n : |n - 2x|/\sqrt{n} \in (j, j+1]\}, \quad y_{n,j} = \max\{x \in E_{n,j} : x \leq n/2\}.$$

Clearly, $[n - (j+1)\sqrt{n}]/2 \leq y_{n,j} < (n - j\sqrt{n})/2$ and

$$E\left(\left|\frac{n - 2\xi_n}{\sqrt{n}}\right|^\theta\right) \leq \sum_{j=0}^{\lfloor \sqrt{n} \rfloor} (j+1)^\theta \pi_n(E_{n,j}). \tag{A.5}$$

Using (5.3), we obtain, for $y_{n,j} \neq 0$,

$$\begin{aligned}\pi_n(E_{n,j}) &= 2^{-n} \sum_{x \in E_{n,j}} \frac{n!}{x!(n-x)!} \leq 2^{1-n} \lceil \sqrt{n} \rceil \frac{n!}{y_{n,j}!(n-y_{n,j})!} \\ &\leq 2^{2-n} \sqrt{n} \beta^3 \frac{n^{n+1/2}}{y_{n,j}^{y_{n,j}+1/2}(n-y_{n,j})^{n-y_{n,j}+1/2}} = 8\beta^3/z_{n,j},\end{aligned}$$

where

$$\begin{aligned} z_{n,j} &= \left(\frac{2}{n}\right)^{n+1} y_{n,j}^{y_{n,j}+1/2}(n-y_{n,j})^{n-y_{n,j}+1/2} \\ &= \left[\frac{2y_{n,j}}{n}\left(2-\frac{2y_{n,j}}{n}\right)\right]^{(n+1)/2}\left(\frac{n-y_{n,j}}{y_{n,j}}\right)^{n/2-y_{n,j}} \\ &= \left[1-\left(1-\frac{2y_{n,j}}{n}\right)^2\right]^{(n+1)/2}\left[\frac{1+(1-2y_{n,j}/n)}{1-(1-2y_{n,j}/n)}\right]^{n/2-y_{n,j}}. \end{aligned}$$

Note that the mapping $t \mapsto (1-t)^{1/t}$ is strictly decreasing on $(0,1)$. This implies

$$\left[1-\left(1-\frac{2y_{n,j}}{n}\right)^2\right]^{n/2} \geq \left[1-\left(1-\frac{2y_{n,j}}{n}\right)\right]^{n/2-y_{n,j}}$$

and, hence,

$$\begin{aligned} z_{n,j} &\geq \sqrt{1-\left(1-\frac{2y_{n,j}}{n}\right)^2}\left[1+\left(1-\frac{2y_{n,j}}{n}\right)\right]^{n/2-y_{n,j}} \\ &\geq \frac{2y_{n,j}}{n}\left[1+\left(1-\frac{2y_{n,j}}{n}\right)\right]^{n/2-y_{n,j}} \end{aligned}$$

In the case $y_{n,j} \geq n/6$, one may use the inequality, $\log(1+t) \geq t/2$ for $t \in [0,1]$, to get

$$z_{n,j} \geq \frac{1}{3}\exp\left\{\frac{n}{4}\left(1-\frac{2y_{n,j}}{n}\right)^2\right\} \geq \frac{1}{3}e^{j^2/4}.$$

In the case $1 \leq y_{n,j} \leq n/6$, it is clear that

$$z_{n,j} \geq \frac{2}{n}\left(\frac{5}{3}\right)^{n/3} \geq \frac{2}{n}e^{n/6} \geq \frac{2}{n}e^{n/24}e^{j^2/8},$$

where the last inequality applies the fact $j < \sqrt{n}$. Putting both cases together, we may choose a universal constant $C > 1$ such that

$$z_{n,j} \geq \frac{e^{j^2/8}}{C}, \quad \forall 0 \leq j \leq \sqrt{n},\ y_{n,j} \neq 0,\ n \geq 1.$$

Back to the computation of $\pi_n(E_{n,j})$, this gives

$$\pi_n(E_{n,j}) \leq 8C\beta^3 e^{-j^2/8}, \quad \forall 0 \leq j \leq \sqrt{n},\ y_{n,j} \neq 0,\ n \geq 1.$$

In fact, the above inequality also holds for $y_{n,j} = 0$ (which must imply $j = \lfloor \sqrt{n} \rfloor$) since, in such a case, $\pi_n(E_{n,j}) = 2^{1-n} \leq 2e^{-(\log 2)j^2} \leq 2e^{-j^2/8}$. Continuing the computation in (A.5), we have

$$
\begin{aligned}
E\left(\left|\frac{n-2\xi_n}{\sqrt{n}}\right|^\theta\right) &\le 8C\beta^3 \sum_{j=0}^{\lfloor\sqrt{n}\rfloor}(j+1)^\theta e^{-j^2/8} \le 16C\beta^3 \sum_{j=0}^{\lfloor\sqrt{n}\rfloor}(j+1)^\theta e^{-(j+2)^2/16} \\
&\le 16C\beta^3 \sum_{j=0}^{\lfloor\sqrt{n}\rfloor}\int_{j+1}^{j+2} t^\theta e^{-t^2/16}dt \le 64C\beta^3 4^\theta \int_0^\infty s^\theta e^{-s^2}ds \\
&= 32C\beta^3 4^\theta \Gamma\left(\frac{\theta+1}{2}\right).
\end{aligned}
$$

□

References

[1] David Aldous. Random walks on finite groups and rapidly mixing Markov chains. In *Seminar on probability, XVII*, volume 986 of *Lecture Notes in Math.*, pages 243–297. Springer, Berlin, 1983.

[2] David Aldous and Persi Diaconis. Shuffling cards and stopping times. *Amer. Math. Monthly*, 93(5):333–348, 1986.

[3] David Aldous and Persi Diaconis. Strong uniform times and finite random walks. *Adv. in Appl. Math.*, 8(1):69–97, 1987.

[4] M. Brown and Y.-S. Shao. Identifying coefficients in the spectral representation for first passage time distributions. *Probab. Engrg. Inform. Sci.*, 1:69–74, 1987.

[5] Guan-Yu Chen. *The cutoff phenomenon for finite Markov chains.* PhD thesis, Cornell University, 2006.

[6] Guan-Yu Chen and Laurent Saloff-Coste. The cutoff phenomenon for ergodic markov processes. *Electron. J. Probab.*, 13:26–78, 2008.

[7] Guan-Yu Chen and Laurent Saloff-Coste. The L^2-cutoff for reversible Markov processes. *J. Funct. Anal.*, 258(7):2246–2315, 2010.

[8] Persi Diaconis and Phil Hanlon. Eigen-analysis for some examples of the Metropolis algorithm. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, volume 138 of *Contemp. Math.*, pages 99–117. Amer. Math. Soc., Providence, RI, 1992.

[9] Persi Diaconis and Laurent Saloff-Coste. Separation cut-offs for birth and death chains. *Ann. Appl. Probab.*, 16(4):2098–2122, 2006.

[10] Jian Ding, Eyal Lubetzky, and Yuval Peres. Total variation cutoff in birth-and-death chains. *Probab. Theory Related Fields*, 146(1-2):61–85, 2010.

[11] William Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons Inc., New York, 1968.

[12] R. Koekoek and R. Swarttouw. The askey-scheme of hypergeometric orthogonal polynomials and its q- analog. http://math.nist.gov/opsf/projects/koekoek.html, 1998.

[13] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times.* American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[14] Laurent Saloff-Coste. Lectures on finite Markov chains. In *Lectures on probability theory and statistics (Saint-Flour, 1996)*, volume 1665 of *Lecture Notes in Math.*, pages 301–413. Springer, Berlin, 1997.

[15] Laurent Saloff-Coste. Random walks on finite groups. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 263–346. Springer, Berlin, 2004.

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